A CHARACTERIZATION OF REGULARITY IN TOPOLOGY

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Abstract. We show in this paper that a topological space satisfies $T_1$ (which we do not intend to imply $T_0$) if and only if convergence of filters is a continuous relation. In particular, a Hausdorff space is regular if and only if convergence of filters is a continuous mapping. We propose a new, categorically motivated, definition of continuous relations between topological spaces, and we compare it with two existing continuity concepts for relations.

Let $(E, \tau)$ be a topological space. We denote by $E^*$ the set of all filters on $E$ which converge for $\tau$ to some point of $E$. For $X \subseteq E$, we put $X^* = \{ \varphi \in E^* : X \subseteq \varphi \}$. Then $\emptyset^* = \emptyset$ for the empty set, and

$$(X \cap Y)^* = X^* \cap Y^*, \quad x \in X^* \iff x \in X,$$

for subsets $X$, $Y$ of $E$, $x \in E$, and $x = \{ X \subseteq E : x \in X \}$. We regard convergence of filters for $\tau$ as a relation $q : E^* \to E$, writing $\varphi \, q \, x$ if $\varphi$ converges to $x$. This relation is a mapping if and only if $(E, \tau)$ is a Hausdorff space. For $X \subseteq E$, we have $q(X^*) = \overline{X}$, the closure of $X$ for $\tau$.

It seems natural to impose a topology on $E^*$ by using the sets $U^*$, with $U$ open for $\tau$, as a basis of open sets. The preceding paragraph shows that this works, and we denote the topology of $E^*$ thus defined by $\tau^*$. With this notation, we state the following theorem.

Theorem 1. A Hausdorff space $(E, \tau)$ is regular if and only if convergence of filters on $E$ for $\tau$ defines a continuous map $q : (E^*, \tau^*) \to (E, \tau)$.

Instead of proving Theorem 1 directly, we generalize it. Theorem 1 is an immediate corollary of Theorem 3 below. We need some definitions.

Let $r : E \to F$ be a relation between two sets. For $X \subseteq E$ and $Y \subseteq F$, we put $y \in r(X)$ if $y \in F$ and $x \, r \, y$ for some $x \in X$, and $x \in r^{-1}(Y)$ if $x \in E$ and $x \, r \, y$ for some $y \in Y$. One sees easily that

$$r(X) \cap Y = \emptyset \iff X \cap r^{-1}(Y) = \emptyset.$$ 

If $E$ and $F$ are topological spaces, then $r$ is called upper semicontinuous.
if $r^{-1}(Y)$ is closed in $E$ for every closed $Y \subseteq F$, and $r$ is called \textit{lower semicontinuous} if $r^{-1}(Y)$ is open in $E$ for every open $Y \subseteq F$. These concepts have been used by various authors; see e.g. [1, Chapter VI] or [3].

A relation $r: E \to F$ between topological spaces has been called continuous if $r$ is both upper and lower semicontinuous. We propose a different definition. We call $r: E \to F$ \textit{continuous} if, for a topological space $A$ and mappings $f: A \to E$ and $g: A \to F$ such that $f(u) \ r \ g(u)$ for all $u \in A$, continuity of $f$ always implies continuity of $g$.

This can be simplified. Let $R \subseteq E \times F$ be the graph of $r$ and $f_1: R \to E$ and $g_1: R \to F$ the projections, i.e. $f_1(x, y) = x$ and $g_1(x, y) = y$ if $x \ r \ y$. Provide $R$ with the coarsest topology for which $f_1$ is continuous. If $r$ is continuous, then $g_1$ is continuous for this topology of $R$. In fact, this is not only necessary but also sufficient for continuity of $r$. For if $f: A \to E$ and $g: A \to F$ are mappings such that $f(u) \ r \ g(u)$ for every $u \in A$, then $f = f_1 h$ and $g = g_1 h$ for a unique mapping $h: A \to R$, and $h$ is continuous, for the given coarse topology of $R$, if $f$ is continuous. Thus continuity of $f$ implies continuity of $g$ if $g_1$ is continuous.

We shall study continuous relations elsewhere in greater detail and in a more general setting. We mention here only that all three continuity concepts defined above coincide with the usual continuity if $r$ is a mapping, and we connect continuity with upper and lower semicontinuity by the following result.

\textbf{Theorem 2.} A continuous relation $r: E \to F$ between topological spaces is upper semicontinuous if and only if its domain $r^{-1}(F)$ is closed in $E$, and $r$ is lower semicontinuous if and only if $r^{-1}(F)$ is open in $E$.

\textbf{Proof.} If $r$ is upper semicontinuous, then $r^{-1}(F)$ is closed in $E$. Conversely, let $R \subseteq E \times F$ be the graph of $r$ and $f_1: R \to E$ and $g_1: R \to F$ the projections, as above. Provide $R$ with the coarsest topology for which $f_1$ is continuous, with the sets $f_1^{-1}(X)$, $X$ closed in $E$, as closed sets. If $r$ is continuous and $Y$ closed in $F$, then $g_1^{-1}(Y)$ is closed in $R$, and thus $g_1^{-1}(Y) = f_1^{-1}(X)$ for a closed set $X \subseteq E$. One sees easily that $r^{-1}(Y) = X \cap r^{-1}(F)$ in this situation. Thus $r^{-1}(Y)$ is closed if $r^{-1}(F)$ is closed. The same argument, with closed sets replaced by open sets, shows that $r$ is lower semicontinuous if and only if $r^{-1}(F)$ is open.\]

The following example shows that Theorem 2 has no obvious converse. For every topological space $E$, the full relation $r: E \to E$ with graph $E \times E$ is both upper and lower semicontinuous. On the other hand, we have $f(u) \ r \ g(u)$ for all $u \in A$ if $f: A \to E$ and $g: A \to E$ are arbitrary mappings. Thus $r$ is continuous only if $E$ is an indiscrete space.
We need one of the separation axioms introduced by Davis [2]. Davis calls a topological space \((E, \tau)\), with filter convergence \(q\), an \(R_0\) space if always \(x \not\approx y \Rightarrow \not\approx y \not\approx x\) for \(x, y \in E\). It is shown in [2] that \(T_1\) is equivalent to the conjunction of \(T_0\) and \(R_0\), and that \(T_3\) (called \(R_2\) in [2]) always implies \(R_0\).

**Theorem 3.** The following three statements are logically equivalent for a topological space \((E, \tau)\) with filter convergence \(q\).

(i) \((E, \tau)\) is a \(T_3\) space.

(ii) \(q: (E^*, \tau^*) \rightarrow (E, \tau)\) is continuous.

(iii) \((E, \tau)\) is an \(R_0\) space and \(q\) is upper semicontinuous.

**Proof.** Assume first \(T_3\) and consider \(f: A \rightarrow E^*\) and \(g: A \rightarrow E\) with \(f\) continuous and \(f(u)\) converging to \(g(u)\) for all \(u \in A\). If \(U\) is open in \(E\) and \(g(u) \in U\), then \(g(u) \in V\) and \(V \subseteq U\) for some open \(V\). For this \(V\), we have \(V \subseteq f(u)\), and \(V \subseteq f(v)\) implies \(g(c) \in V\). Thus \(u \in f^{-1}(V^*)\) and \(f^{-1}(V^*) \subseteq g^{-1}(U)\). This shows that \(g^{-1}(U)\) is open, and hence \(g\) continuous.

If \(q\) is continuous, then \(q\) is upper semicontinuous by Theorem 2. If \(x \not\approx y\), let \(A\) be the space with two points \(u, v\), and with \(\{v\}\) open, but not closed. Put \(f(u) = f(v) = x\) and \(g(u) = x\), \(g(v) = y\). Then \(f\) is continuous, and \(f(z) \not\approx g(z)\) for \(z \in A\). Thus \(q\) is continuous. If \(U\) is open and \(x \in V\), then \(g^{-1}(V)\) is open and \(u \in g^{-1}(V)\). Thus \(g^{-1}(V) = A\), and \(y \in V\). This shows that also \(y \not\approx x\), and \(E\) is \(R_0\).

Assume now (iii), and let \(F\) be closed in \(E\) and \(x \in E \setminus F\). If \(x \not\approx y\), then \(y \not\approx x\), and \(y \in F\) would imply \(x \in F = F\). Thus \(x \not\approx g^{-1}(F)\). It follows that \(x \in V^*\) for an open set \(V\) with \(V^* \cap q^{-1}(F) = \emptyset\). But then \(x \in V\), and \(V \cap F = q(V^*) \cap F = \emptyset\). Thus \(E\) satisfies \(T_3\).

The following example shows that \(R_0\) cannot be omitted from Theorem 3. The space with two points and three open sets (used in the proof of the theorem) is \(T_0\) but not \(T_1\), and hence a fortiori not \(T_3\) or \(R_0\). But one sees easily that \(q\) is upper semicontinuous for this space.

**Remark.** All results of this note remain valid if \(E^*\) is replaced by a set of convergent filters which contains all convergent ultrafilters.

**References**


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