WALLMAN-TYPE COMPACTIFICATIONS AND PRODUCTS

FRANK KOST

Abstract. Y is a Wallman-type compactification (O. Frink, Amer. J. Math. 86 (1964), 602-607) of X in case there is a normal base Z for the closed sets of X such that the ultrafilter space from Z, denoted ω(Z), is topologically Y. It is not known if every compactification is Wallman-type. For Za a normal base for the closed sets of Xa for each α belonging to an index set Λ it is shown that the Tychonoff product space Π α∈Λ ω(Zα) is a Wallman compactification of Π α∈Λ Xα. Also for X⊂T⊂ω(Z) with Z a normal base for the closed sets of X, a proof that ω(Z) is a Wallman-type compactification of T is indicated.

Introduction. In 1938, H. Wallman [5] associated with a T1-space X a compact T1-space S which contained a dense copy of X. In this construction the points of S are the ultrafilters from the lattice of closed subsets of X. O. Frink [2] characterized complete regularity in T1-spaces as those that possess a normal base for their closed sets. In the proof of sufficiency he constructed a Hausdorff compactification ω(Z) of X by considering the ultrafilters from the normal base Z. As this construction was utilized by Wallman, Frink called a compactification obtained in this way a Wallman-type compactification and offered the conjecture that every compactification is Wallman-type. This question is unsettled.

The Stone-Čech compactification of X [3] and the one-point compactification of locally compact X [1] are Wallman-type. Sufficient conditions on a compact space are known that insure that it is a Wallman-type compactification of each of its dense subspaces [4]. Every well-ordered set with the last element endowed with the order topology enjoys these conditions. Hence the set of ordinal numbers less than or equal to the first uncountable one provides an example of a Wallman-type compactification of an uncountable discrete space (the nonlimit ordinals).

It is natural to ask if the Tychonoff product space Π α∈Λ ω(Zα) is a Wallman-type compactification of Π α∈Λ Xα where ω(Zα) is a Wallman-
type compactification of \( X_\alpha \) for each \( \alpha \in \Delta \) and \( \Delta \) an index set of arbitrary cardinal. We show that it is.

**Definitions.** \( Y \) is a compactification of \( X \) if \( Y \) is a compact Hausdorff space and a copy of \( X \) is dense in \( Y \). A family \( Z \) of closed sets of a space \( X \) is said to be

(i) a base for the closed sets of \( X \) if \( x \in A \), \( A \) closed, there is \( F \in Z \) with \( x \in F \) and \( A \subseteq F \);

(ii) a ring of sets if \( Z \) is closed under the formation of finite intersections and unions;

(iii) disjunctive if for \( x \in A \), \( A \) closed, there is \( F \in Z \) with \( x \in F \) and \( F \cap A = \emptyset \);

(iv) a normal family if, whenever \( F_1, F_2 \in Z \) with \( F_1 \cap F_2 = \emptyset \), there is \( H_1, H_2 \in Z \) with \( F_1 \subseteq X \setminus H_1 \), \( F_2 \subseteq X \setminus H_2 \) and \( (X \setminus H_1) \cap (X \setminus H_2) = \emptyset \), i.e., complements of members of \( Z \) separate disjoint members of \( Z \).

In case \( Z \) satisfies (i)\( \rightarrow \) (iv) above \( Z \) is called a normal base for the closed sets of \( X \). The lattice of all closed sets of a \( T_1 \)-space \( X \) possesses (i)\( \rightarrow \) (iii) and is a normal base if and only if \( X \) is a normal space. The collection of finite unions of sets of the form \( (0, b], [a, 1), [a, b] \) with \( 0 < a, b < 1 \) is a normal base for the closed sets of \( (0, 1) \) with the usual topology. For \( X \) a discrete space we have \( \{ F \subseteq X : F \) finite or \( X \setminus F \) finite} is a normal base. In case \( X \) is completely regular-\( T_1 \) (\( T_{\text{reg}} \)) the family of zero-sets of continuous real-valued functions on \( X \) is a normal base. For \( Z \) a ring of closed sets that is a disjunctive base for the closed sets of \( X \) we set \( \omega(Z) \equiv \{ \alpha : \alpha \subseteq Z, \alpha \) an ultrafilter\}. The ultrafilters from \( Z \) with non-empty intersection correspond to the points of \( X \) and those with empty intersection (free ultrafilters) are the points that are “added” to \( X \) to obtain \( \omega(Z) \). If \( X \) is compact there are no free ultrafilters and no points to be “added.” \( \omega(Z) \) is topologized as follows: for \( F \in Z \) define \( F^* \equiv \{ \alpha \in \omega(Z) : F \in \alpha \} \). As \( Z \) is closed under finite unions, \( \{ F^* : F \in Z \} \) is a base for the closed sets of some topology on \( \omega(Z) \). \( \omega(Z) \), with this structure, is the compact \( T_1 \)-space of our concern.

**Lemma 1.** Let \( Z \) be a ring of closed sets that is a disjunctive base for the closed sets of \( X \). \( \omega(Z) \) is a Hausdorff space if and only if \( Z \) is a normal family.

**Proof.** The sufficiency appears in [2] and is basic to the conjecture. Now assume \( \omega(Z) \) is Hausdorff. For \( F_1, F_2 \in Z \) with \( F_1, F_2 \) disjoint, it follows that \( F_1^* \cap F_2^* = \emptyset \). \( \omega(Z) \) Hausdorff implies \( \omega(Z) \) is normal since it is also compact. As a result \( F_1^* \) and \( F_2^* \) can be separated by
basic open sets $O_1$ and $O_2$ in $\omega(Z)$. The traces of $O_1$ and $O_2$ in $X$ separate $F_1$ and $F_2$ and their complements belong to $Z$.

The collection of finite unions from $(0, b], [a, 1], [a, b]$ with $0 < a, b < 1$ yields the 2-point compactification $([0, 1])$ of $(0, 1)$. For $X$ discrete and $Z = \{ F \subseteq X : F$ finite or $X \setminus F$ finite $\}$, the space $\omega(Z)$ is the one-point compactification of $X$, i.e., there is just one free ultrafilter from $Z$. If $X$ is $T_4$, $Z$ is the zero sets of real valued continuous functions on $X$, Gillman and Jerison [3] have shown that $\omega(Z) = \beta(X)$ where $\beta X$ is the Stone-Cech compactification of $X$. Frink's conjecture can be rephrased as follows: If $Y$ is a compactification of $X$ does there exist a normal base $Z$ for the closed sets of $X$ with $\omega(Z) \approx Y$.

For $Z_a$ a normal base for the closed sets of $X_a$, for each $\alpha \in \Delta$, we can form $\omega(Z_a)$. The cartesian product space $\prod_{\alpha \in \Delta} \omega(Z_a)$ is a compactification of $\prod_{\alpha \in \Delta} X_a$ which, as we proceed to show, is Wallman-type. First we need a lemma.

**Lemma 2.** Take $\mathcal{F}$ a family of sets closed under finite intersection. There is a 1-1 correspondence between the ultrafilters from $\mathcal{F}$ and those from $\mathcal{F}_x$ where $\mathcal{F}_x$ is all finite unions from $\mathcal{F}$.

**Proof.** The correspondence we obtain will be $\alpha \rightarrow \mathcal{F}_\alpha$ where $\mathcal{F}$ is an ultrafilter from $\mathcal{F}$ and $\mathcal{F}_\alpha$ is the unique ultrafilter from $\mathcal{F}_x$ that contains $\alpha$. First note that distinct ultrafilters from $\mathcal{F}$ can be embedded in distinct ultrafilters from $\mathcal{F}_x$. Now let $\mathcal{F}$ be an ultrafilter from $\mathcal{F}_x$. $\mathcal{F}$ is $\{ F_\alpha : \alpha \in \Delta \}$ with $F_\alpha = \bigcup_{i \in I_\alpha} A_\alpha$ for each $\alpha \in \Delta$. $\mathcal{F}$ maximal implies $\mathcal{F}$ prime so for some $i_0$, $1 \leq i_0 \leq n$, $A_{\alpha_{i_0}} \in \mathcal{F}$. Denote $A_{\alpha_{i_0}}$ by $A_\alpha$ and observe that $\{ A_\alpha \} \subseteq \mathcal{A}$ has the Finite Intersection Property in $\mathcal{F}$, so it can be embedded in an ultrafilter $\alpha$ from $\mathcal{F}$. If $A \in \alpha$ then $A \cap A_\alpha \neq \emptyset$ for every $\alpha \in \Delta$ and since $A \subseteq F_\alpha$ we have $A \cap F_\alpha \neq \emptyset$ for every $\alpha \in \Delta$. Since $A$ intersects every member of $\mathcal{F}$ it follows that $A \subseteq \mathcal{F}$ and $\alpha \subseteq \mathcal{F}$. To see uniqueness let $\mathcal{F}_1$ and $\mathcal{F}_2$ be ultrafilters from $\mathcal{F}_x$ each containing $\alpha$. Assume $\mathcal{F}_1 \neq \mathcal{F}_2$ and take $F \in \mathcal{F}_2$ with $F \subseteq \mathcal{F}_1$. $F = \bigcup_{i=1}^n A_i, A_i \in \mathcal{F}$ and $\mathcal{F}_2$ prime implies there is $i_0, 1 \leq i_0 \leq n$, with $A_{i_0} \in \mathcal{F}_2$. Now $A_{i_0}$ intersects every member of $\alpha$ $[\alpha \subseteq \mathcal{F}_2]$ so we have $A_{i_0} \subseteq \alpha$. There is $F_1 \subseteq \mathcal{F}_1$ with $F_1 \cap F = \emptyset$ and since $A_{i_0} \subseteq F$ it follows that $A_{i_0} \cap F_1 \neq \emptyset$, i.e., $A_{i_0} \subseteq \mathcal{F}_1$ which contradicts our assumption that $\alpha \subseteq \mathcal{F}_2$. Therefore $\mathcal{F}_1 = \mathcal{F}_2$. In fact, $\mathcal{F}_1 \cap \mathcal{F}_2 = \emptyset$.

**Theorem.** Let $Z_a$ be a normal base for the closed sets of $X_a$ for each $\alpha \in \Delta$. $\prod_{\alpha \in \Delta} \omega(Z_a)$ is a Wallman-type compactification of $\prod_{\alpha \in \Delta} X_a$.

**Proof.** Let $Z = \{ \prod F_\alpha : F_\alpha \in Z_a \text{ and } F_\alpha = X_a \text{ for all but a finite number of } \alpha \in \Delta \}$ and $Z_\mathcal{F}$ be the collection of all finite unions from $Z$. References [3].
The family $C$ of complements of members of $Z_j$ is a collection of open subsets of $\prod X_a$ that contains $\{\prod_{\alpha \in \Delta} O_{\alpha} : O_{\alpha} = X_{\alpha} \text{ for all but finitely many } \alpha, X_{\alpha} \setminus O_{\alpha} \in Z_{\alpha} \text{ for other } \alpha\}$. Hence $C$ is a base for the open sets of $\prod X_a$ and $Z_j$ is a base for the closed sets of $\prod X_a$. To see that $Z_j$ is disjunctive take $x = (x_{\alpha}) \in A$, $A$ closed in $\prod X_a$.

Since $Z_j$ is a base there is $F \in Z_j$ with $(x_{\alpha}) \in F$ and $A \subseteq F$. Now $F = \bigcup_{i=1}^n A_i, A_i \subseteq Z_j$ and for each $i, 1 \leq i \leq n$, $x = (x_{\alpha}) \in A_i = \bigcap_{\alpha \in A} A_i$. So there is $\alpha \in \Delta$ with $x_{\alpha} \in A_i$, $Z_{\alpha}$ is disjunctive so take $F_\alpha \in Z_{\alpha}$ with $x_{\alpha} \in F_\alpha$ and $F_\alpha \cap A_i = \emptyset$. Let $B_i$ be the set $\bigcap_{\alpha \in A} A_i$ with $F_\alpha$ replacing $A_i$. For each $i, x = (x_{\alpha}) \in B_i$ and $B_i \cap A_i = \emptyset$ so we have $x \in \bigcap_{i=1}^n B_i \in Z_j$ and $(\bigcap_{i=1}^n B_i) \cap F = \emptyset$. That $Z_j$ is a ring of sets results from $Z_j$ being closed under finite intersections and from the finite distributive laws of sets. We have that $Z_j$ is a disjunctive base for the closed sets of $\prod X_a$ and a ring of sets, so the ultrafilter space $\omega(Z_j)$ can be formed. To continue the proof we need

Lemmas 3. There is a 1-1 correspondence between points in $\prod_{\alpha \in \Delta} \omega(Z_{\alpha})$ and ultrafilters from $Z$.

Proof. Take $(a_{\alpha}) \in \prod_{\alpha \in \Delta} \omega(Z_{\alpha})$ and define $\prod a_{\alpha} = \{\prod A_{\alpha} : A_{\alpha} \subseteq a_{\alpha} \text{ and } A_{\alpha} = X_{\alpha} \text{ for all but finitely many } \alpha \in \Delta\}$. It is easily seen that $\prod a_{\alpha}$ is a filter from $Z$ and $(a_{\alpha}) \neq (a_{\beta})$ implies $\prod a_{\alpha} \neq \prod a_{\beta}$. Now let $3$ be an ultrafilter from $Z$ and for $\gamma \in \Delta$ set $3_\gamma = \{F_{\gamma} = F\}$, i.e., $3_\gamma$ is the collection of $\gamma$th components of members of $3$. That $3_\gamma$ is a filter from $Z_{\gamma}$ is a consequence of the equality $(\prod A_{\alpha}) \cap (\prod B_{\gamma}) = \prod (A_{\alpha} \cap B_{\gamma})$ for $A_{\alpha}, B_{\gamma}$ subsets of $X_{\alpha}$. To see that $3_\gamma$ is maximal take a filter $\alpha$ from $Z_{\gamma}$ with $3_{\gamma} \subseteq \alpha$. Take $A \subseteq \alpha$ and let $\prod F_{\alpha}$ be any member of $3$ and define $F' = \prod F_{\alpha}$ where $F_{\gamma} = A$. We wish to show that $F'$ meets every member of $3$ in which case we have $F' \subseteq F_3$ and $F_{\gamma} = A \subseteq 3_\gamma$ and $3_\gamma$ is an ultrafilter. Let $\prod A_{\alpha} \subseteq F'$, $F' \cap (\prod A_{\alpha}) = \prod (F_{\alpha} \cap A_{\alpha})$ where $F_{\gamma} = A$. Now $A \cap A_{\alpha} \neq \emptyset$ since $A_{\gamma} \subseteq 3_\gamma \subseteq \alpha$ and $\alpha$ is a filter. We have $(\prod F_{\alpha}) \cap (\prod A_{\alpha}) \neq \emptyset$ as both belong to $3$ so $F_{\alpha} \cap A_{\alpha} \neq \emptyset$ for $\alpha \neq \gamma$. As a result $F' \cap (\prod A_{\alpha}) \neq \emptyset$. For $3$ an ultrafilter from $Z$ we have $3_{\gamma}$ an ultrafilter from $Z_{\gamma}$ for each $\gamma \in \Delta$ and the filter $\prod_{\gamma \in \Delta} 3_{\gamma}$ from $Z$ contains $3$. Therefore $\prod_{\alpha \in \Delta} 3_{\gamma} = 3$ and the ultrafilters from $Z$ are precisely $\{\prod_{\alpha \in \Delta} a_{\alpha} \mid (a_{\alpha}) \in \prod_{\alpha \in \Delta} \omega(Z_{\alpha})\}$.

Define $\theta : \prod \omega(Z_{\alpha}) -+ \omega(Z_j)$ by $\theta((a_{\alpha})) = 3_{\prod a_{\alpha}}$ where $3_{\prod a_{\alpha}}$ is the ultrafilter from $Z_j$ that contains the ultrafilter $\prod a_{\alpha}$ from $Z_j$. By Lemmas 2 and 3, $\theta$ is well defined, 1-1, and onto. To complete the proof of Theorem 1 we need a final lemma.

Lemmas 4. $\theta$ is a homeomorphism.
Proof. To establish continuity of \( \theta \) we take a basic closed set \( F^* \subseteq \omega(Z) \) and show that \( \theta^{-1}(F^*) \) is closed in \( \prod \omega(Z) \). Now \( F \subseteq Z \) so \( F = \bigcup_{i=1}^{n} F_i, F_i \subseteq Z, \) and note that \( F^* = \bigcup_{i=1}^{m} (F_i^*) \). Recall that \( F^* \) is the collection of ultrafilters from \( Z \) that contain \( F \). Set \( F_i = \prod_{a \in A} A_i \) and \( \theta^{-1}(F^*) = \theta^{-1}(\bigcup_{i=1}^{m} (F_i^*)) = \bigcup_{i=1}^{m} \theta^{-1}(F_i^*) \). We wish to show \( \theta^{-1}(F_i^*) = \prod \langle A_i \rangle \) which is a basic closed set in \( \prod \omega(Z) \).

If \( \langle \alpha \rangle = (\alpha) \) then \( \theta(\langle \alpha \rangle) = \langle \alpha \rangle \), which means \( \langle \alpha \rangle \in \theta^{-1}(F_i^*) \). Therefore \( \langle \alpha \rangle \in \prod \langle A_i \rangle \) and \( \theta^{-1}(F_i^*) \subseteq \prod \langle A_i \rangle \). If \( \langle \alpha \rangle \in \prod \langle A_i \rangle \) then \( \alpha \in \bigcup_{i=1}^{m} \langle A_i \rangle \) for each \( \alpha \in A \) and \( A_i \subseteq \alpha \) so \( \prod A_i \subseteq \prod \alpha \). This implies \( \prod A_i \subseteq \bigcup_{i=1}^{m} \langle A_i \rangle \). However \( \bigcup_{i=1}^{m} \langle A_i \rangle = \theta^{-1}(F^*) \) and we have \( \theta^{-1}(F^*) \subseteq \theta^{-1}(F_i^*) \). Therefore \( \prod \langle A_i \rangle \subseteq \theta^{-1}(F_i^*) \) and \( \theta^{-1}(F_i^*) = \prod \langle A_i \rangle \). As a result \( \theta^{-1}(F^*) \) is a closed set in \( \prod \omega(Z) \).

By applying \( \theta \) to both sides of \( \theta^{-1}(F^*) = \prod \langle A_i \rangle \) we obtain \( \theta^{-1}(F_i^*) = \prod \langle A_i \rangle \) and \( \theta \) takes basic closed sets in \( \prod \omega(Z) \) to closed sets in \( \omega(Z) \), i.e., \( \theta \) is a closed map. We have established that \( \theta \) is a homeomorphism so \( \omega(Z) \) is Hausdorff since \( \prod \omega(Z) \) is.

By Lemma 1, \( Z \subseteq \omega(Z) \) is a normal base for \( \prod X \) and as a result \( \prod \omega(Z) \) is a Wallman-type compactification of \( \prod X \). This completes the proof of the theorem.

By taking finite products of Wallman-type compactifications of \( N \), the countably infinite discrete space, a class of compactifications of \( N \) are obtained. By the theorem each is Wallman-type. Similarly for any discrete \( X \). Also \( \prod X \subseteq X \) is a Wallman-type compactification of \( \prod X \).

For \( \omega(Z) \) a Wallman compactification of \( X \), \( \{ F^* : F \subseteq Z \} \) is a normal base for the closed sets of \( \omega(Z) \) and for \( X \subseteq T \subseteq \omega(Z) \) the collection \( T \subseteq \omega(Z) \) is a normal base for the closed sets of \( T \). For \( p \subseteq \omega(Z) \) define \( A_p = \{ F^* \subseteq T : p \subseteq F^* \} \). The ultrafilters from \( T \) are precisely \( \{ A_p : p \subseteq \omega(Z) \} \). The free ultrafilters are those \( A_p \) with \( p \subseteq T \). The mapping \( \theta : \omega(Z) \to \omega(Z) \) by \( \theta(p) = A_p \) is a homeomorphism and \( \omega(Z) \) is a Wallman-type compactification of \( T \). Therefore \( \prod \omega(Z) \) is a Wallman compactification of \( T \) for \( \prod X \subseteq T \subseteq \omega(Z) \).

As mentioned earlier the conjecture that every compactification is Wallman-type is unsettled. It is not known if all compactifications of \( N \) are. We have removed a class of the pathological Tychonoff product spaces as candidates for a counterexample. The problem of finding a counterexample seems difficult as one would have to prove the non-existence of an appropriate normal base.
References


State University College, Oneonta, New York 13820