UNIQUENESS OF SOLUTIONS OF CERTAIN BOUNDARY VALUE PROBLEMS FOR ULTRAHYPERBOLIC EQUATIONS

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1. Introduction. In [1], Owens has given sufficient conditions for uniqueness of solutions of the ultrahyperbolic differential equation

\[ \sum_{i=1}^{m} u_{x_ix_i} - \sum_{j=1}^{n} u_{y_jy_j} = 0 \quad (m \geq 2, n \geq 2) \]

for certain mixed problems having elliptic and hyperbolic nature. However, problems of the Dirichlet and Neumann type are not considered. The purpose of this note is to present necessary and sufficient conditions for the uniqueness of solutions of the Dirichlet and Neumann problems for the more general ultrahyperbolic differential equation

\[ Lu = \sum_{i=1}^{m} u_{x_ix_i} - \sum_{j,k=1}^{n} (a_{jk} u_{y_j}) y_k + cu = 0 \]

in a domain \( Q = X \times Y \), where \( X \) is a hyper-parallelepiped defined by \( 0 < x_i < a_i, 1 \leq i \leq m \), and \( Y \) is a bounded domain in the \( n \)-dimensional space \( y_1, \ldots, y_n \). The procedure we shall use here is an extension of that employed in [2], [3], [4], and [5] for the normal hyperbolic equations. Since our treatment remains valid for \( m \geq 1 \), our results here, therefore, contain also those obtained in [2], [3], and [4].

Throughout this paper, we assume that the coefficients \( a_{jk} \) and \( c \) depend only on the variables \( y_1, \ldots, y_n \), and are continuous functions of these variables with \( c \geq 0 \) in \( Y \). Moreover, we assume that the coefficient matrix \( (a_{jk}) \) is symmetric and positive definite, and that \( a_{jk} \), together with the boundary \( \partial Y \) of \( Y \), are sufficiently smooth in order to ensure the existence of complete sets of eigenfunctions for the eigenvalue problems that we will need below.

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For brevity, we shall let \( x \) and \( y \) stand for the sets of variables \((x_1, \cdots, x_m)\) and \((y_1, \cdots, y_n)\), respectively, and write \( u(x, y) \) for 
\( u(x_1, \cdots, x_m, y_1, \cdots, y_n) \).

2. The Dirichlet problem. It suffices to consider the homogeneous problem

\[
Lu = 0 \quad \text{in } Q, \quad u = 0 \quad \text{on } \partial Q.
\]

**Theorem 1.** Let \( \lambda_k (k = 1, 2, \cdots) \) be the eigenvalues of the problem

\[
\sum_{i,k=1}^{n} (a_{ik}v_i)v_k - cv + \lambda v = 0 \quad \text{in } Y, \quad v = 0 \quad \text{on } \partial Y.
\]

Then every solution \( u \in C^2(Q) \cap C^1(\overline{Q}) \) of the problem (3) vanishes identically in \( Q \) if, and only if,

\[
\pi^2 \sum_{i=1}^{m} (k_i/a_{i})^2 \neq \lambda_k
\]

for all nonzero integers \( k_1, \cdots, k_m \).

**Proof.** The necessity part follows easily. Indeed, suppose there exist nonzero integers \( p_1, \cdots, p_m \) and an eigenvalue \( \lambda_p \) of (4) such that

\[
\pi^2 \sum_{i=1}^{m} (p_i/a_{i})^2 = \lambda_p.
\]

Let \( v_p \) be an eigenfunction of (4) corresponding to \( \lambda_p \). Then the function

\[
u(x, y) = \prod_{i=1}^{m} \sin \left( \frac{p_i \pi}{a_i} x_i \right) v_p(y)
\]

is a nontrivial solution of the problem (3), as is readily verified.

Conversely, suppose that condition (5) holds. By the divergence theorem, we have

\[
\int_{Q} (uLw - wLu) dx \ dy = \int_{\partial Q} \left[ \sum_{i=1}^{m} (uw_{x_i} - u_{x_i}w)v_{x_i} - \sum_{j,k=1}^{n} a_{jk}(uw_{y_j} - u_{y_j}w)v_{y_k} \right] dS
\]

where \( dx = dx_1 \cdots dx_m, dy = dy_1 \cdots dy_n \), \( dS \) is the surface element on \( \partial Q \), and \( v_{x_i} \) and \( v_{y_k} \) are the direction cosines of the outward normal.
vector on $\partial X$ and $\partial Y$ respectively. Now let $u$ be a solution of (3) and choose

$$w(x, y) = \prod_{i=1}^{m-1} \left( \sin \frac{k_i \pi}{a_i} x_i \right) (\sin \mu_m x_m) v_k(y)$$

where the nonzero integers $k_1, \cdots, k_{m-1}$ and the nonzero constant $\mu_m$ satisfy the relation

$$\pi^2 \sum_{i=1}^{m-1} \left( \frac{k_i}{a_i} \right)^2 + \mu_m^2 = \lambda_k$$

with $v_k$ being an eigenfunction of (4) corresponding to the eigenvalue $\lambda_k$. Notice that $\mu_m$ may very well be an imaginary number. Then, as is readily seen, the function (8) satisfies $Lw = 0$ and vanishes on $\partial Q$, except on $x_m = a_m$ in view of condition (5). Substitution of these functions into the formula (7) thus yields

$$\int \left\{ \int_{X'} u_{x_m}(x', a_m, y) \prod_{i=1}^{m-1} \left( \sin \frac{k_i \pi}{a_i} x_i \right) dx' \right\} v_k(y) dy = 0$$

for all eigenfunctions $v_k$ and for all nonzero integers $k_1, \cdots, k_{m-1}$. Here $x' = (x_1, \cdots, x_{m-1})$, $dx' = dx_1 \cdots dx_{m-1}$, and $X'$ denotes the subspace $x_1, \cdots, x_{m-1}$ of $X$. Since the set of eigenfunctions $\{v_k\}$ is complete, equation (10) states that

$$\int_{X'} u_{x_m}(x', a_m, y) \prod_{i=1}^{m-1} \left( \sin \frac{k_i \pi}{a_i} x_i \right) dx' = 0.$$

This, in turn, implies that

$$u_{x_m}(x', a_m, y) = 0$$

in view of the completeness of the set of eigenfunctions

$$\left\{ \prod_{i=1}^{m-1} \sin \frac{k_i \pi}{a_i} x_i \right\} \quad \text{in} \quad X'.$$

Now let us integrate the differential identity

$$(2x_m u_{x_m} + u) L u$$

over $Q$ and use the divergence theorem. Since $u = 0$ on $\partial Q$ and $u_{x_m} = 0$
on $x_m = a_m$, all the surface integrals resulting from (12) vanish and we have

$$\int \int_Q (2x_m u x_m + u) Lu\, dx\, dy = -2 \int \int_Q u x_m^2\, dx\, dy.$$  

Since $Lu = 0$, this implies that $u$ is independent of $x_m$. But $u = 0$ on $x_m = a_m$, hence $u = 0$ in $Q$.

Our proof of Theorem 1 shows that, whenever condition (5) holds, it is not necessary to prescribe the values of the normal derivative of $u$ on a face of $X$, as required in Theorem 2 of [1].

3. The Neumann problem. Consider next the homogeneous Neumann problem

$$Lu = 0 \quad \text{in } Q, \quad \frac{\partial u}{\partial n} = 0 \quad \text{on } \partial Q;$$

where on $X \times \partial Y$, the derivative $\frac{\partial u}{\partial n}$ is defined by

$$\frac{\partial u}{\partial n} = \sum_{j,k=1}^n a_{jk} u_{y_j} v_{y_k}.$$  

When the quantities $\lambda_k$ are taken to be eigenvalues of a corresponding boundary value problem of the Neumann type, it turns out that condition (5) is also necessary and sufficient for $u = 0$ to be the only solution of the problem (14), provided $\epsilon \neq 0$.

Theorem 2. Let $\lambda_k$ be the nonzero eigenvalues of the problem

$$\sum_{j,k=1}^n (a_{jk} v_{y_j})_{y_k} - cv + \lambda v = 0 \quad \text{in } Y, \quad \frac{\partial v}{\partial n} = 0 \quad \text{on } \partial Y.$$  

Then every solution $u \in C^2(Q) \cap C^1(\overline{Q})$ of (14) vanishes identically (or $u = \text{const}$ in case $\epsilon = 0$) if, and only if,

$$\pi^2 \sum_{i=1}^m \frac{(k_i/a_i)^2}{\lambda_k} \neq \lambda_k$$

for all integers $k_1, \ldots, k_m$.

Proof. If there exist integers $p_1, \ldots, p_m$, and a nonzero eigenvalue $\lambda_p$ of (15), for which (16) does not hold, then a nonconstant and hence a nontrivial solution of (14) is given by

$$u(x, y) = \prod_{i=1}^m \left( \cos \frac{k_i \pi}{a_i} x_i \right) v_p(y)$$
where \( v_p \) is an eigenfunction corresponding to \( \lambda_p \).

On the other hand, suppose that condition (16) holds. Let \( u \) be a solution of (14) and choose

\[
 w(x, y) = \prod_{i=1}^{m-1} \left( \cos \frac{k_i \pi}{a_i} x_i \right) \left( \cos \mu_m x_m \right) v_k(y)
\]

where the integers \( k_1, \ldots, k_{m-1} \) and the constant \( \mu_m \) satisfy the relation (9). Substituting these functions in the formula (7), and using condition (16), we then obtain

\[
 \int_Y \left\{ \int_{X'} u(x', a_m, y) \prod_{i=1}^{m-1} \left( \cos \frac{k_i \pi}{a_i} x_i \right) dx' \right\} v_k(y) dy = 0
\]

for all integers \( k_1, \ldots, k_{m-1} \), and for all eigenfunctions \( v_k \) corresponding to the nonzero eigenvalues \( \lambda_k \). By the completeness of the set \( \{ v_k \} \), equation (18) implies that

\[
 \int_{X'} u(x', a_m, y) \prod_{i=1}^{m-1} \left( \cos \frac{k_i \pi}{a_i} x_i \right) dx' = \nu = \text{const.}
\]

with \( \nu \equiv 0 \) in case \( c > 0 \). Notice that when \( c \equiv 0 \), \( v_0 = 1 \) is an eigenfunction of (15) corresponding to the eigenvalue \( \lambda_0 = 0 \). Now, since the set of eigenfunctions \( \left\{ \prod_{i=1}^{m-1} \cos \left( \frac{k_i \pi}{a_i} x_i \right) \right\} \) is complete in \( X' \), equation (19) in turn implies that \( u(x', a_m, y) = \text{const.} \), which is zero if \( c > 0 \).

Let us consider the case \( c > 0 \). By integrating the identity (12) over \( Q \) and using the fact \( u = 0 \) on \( x_m = a_m \), we again arrive at equation (13) from which the conclusion that \( u \equiv 0 \) in \( Q \) follows.

In case \( c = 0 \), the above argument yields the result \( u = \text{const.} \) in \( Q \).

4. Mixed boundary value problems. The method we have used to establish Theorems 1 and 2 above can also be employed to obtain uniqueness results for equation (1) with mixed boundary conditions of the following type:

\[
 u = 0 \quad \text{on} \quad \partial X \times \bar{Y}, \quad \frac{\partial u}{\partial n} = 0 \quad \text{on} \quad \bar{X} \times \partial Y;
\]

\[
 u = 0 \quad \text{on} \quad x_i = 0, \quad x_i = a_i \quad (1 \leq i \leq p),
\]

\[
 u_{x_j} = 0 \quad \text{on} \quad x_j = 0, \quad x_j = a_j \quad (p + 1 \leq j \leq m),
\]

\[
 \frac{\partial u}{\partial n} = 0 \quad \text{on} \quad \bar{X} \times \partial Y;
\]

\[
 \frac{\partial u}{\partial n} = 0 \quad \text{on} \quad \partial X \times \bar{Y}, \quad u = 0 \quad \text{on} \quad \bar{X} \times \partial Y;
\]
\[ u = 0 \text{ on } x_i = 0, x_i = a_i \quad (1 \leq i \leq \rho), \]
\[ u_{xj} = 0 \text{ on } x_j = 0, x_j = a_j \quad (\rho + 1 \leq j \leq m), \]
\[ u = 0 \text{ on } \bar{X} \times \partial V. \]

Specifically, we have the following uniqueness theorems corresponding to each of these boundary conditions.

**Theorem 3.** Every solution \( u \in C^2(Q) \cap C^1(\bar{Q}) \) of equation (2) in \( Q \) satisfying the boundary conditions (20) [or (21)], vanishes identically in \( Q \) if, and only if, (5) holds for all nonzero integers \( k_1, \ldots, k_m \) [or nonzero integers \( k_1, \ldots, k_p \) and integers \( k_{p+1}, \ldots, k_m \)], where \( \lambda_k \) \((k = 1, \ldots, m)\) are the eigenvalues of the problem (15).

**Theorem 4.** Every solution \( u \in C^2(Q) \cap C^1(\bar{Q}) \) of equation (2) in \( Q \) satisfying the boundary conditions (22) [or (23)], vanishes identically in \( Q \) if, and only if, (5) holds for all integers \( k_1, \ldots, k_m \) [or nonzero integers \( k_1, \ldots, k_p \) and integers \( k_{p+1}, \ldots, k_m \)], where \( \lambda_k \) are the eigenvalues of the problem (4).

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**References**


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