CERTAIN NUMERICAL RADIUS CONTRACTION OPERATORS

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Abstract. In this paper an operator $T$ means a bounded linear operator on a complex Hilbert space $H$. The numerical radius norm $w(T)$ of an operator $T$, is defined by $w(T) = \sup \{ |(Tx, x)| \} \text{ for every unit vector } x \in H$. An operator $T$ is said to be a numerical radius contraction if $w(T) \leq 1$. We shall give some theorems on certain numerical radius contraction operators and related results in consequence of these theorems.

Our central result is that an idempotent numerical radius contraction is a projection. Finally we prove that a periodic numerical radius contraction is the direct sum of zero and a unitary operator, that is to say, normal and partial isometric.

We shall begin with the following lemma and its consequence.

Lemma. If $T$ is idempotent and $w(T) \leq 1$, then $T$ is a projection.

We shall give two proofs of the Lemma as follows.

First proof of the Lemma. Let $y = Tx$ where $x \in R(T)^\perp$. Then for any positive number $t$ we have $T(x + ty) = (1 + t)y$ so that

$$
(1 + t)||y||^2 = \langle (1 + t)y, x + ty \rangle = \langle T(x + ty), x + ty \rangle
$$

$$
\leq ||x + ty||^2 = ||x||^2 + t^2||y||^2.
$$

Hence $t||y||^2 \leq ||x||^2$ so that $y = 0$. Therefore $T = 0$ on $R(T)^\perp$ and $T = 1$ on $R(T)$ so that $T$ is a projection.

To give the second proof of the Lemma we need the following unpublished paper [8] which is cited with Professor Kato's permission.

Theorem A. The following two conditions are equivalent

(i) $w(T) \leq 1$,

(ii) $||e^{it\theta}|| \leq e^{t|\theta|}$ for all complex numbers $\theta$.

Proof. Here we cite the proof of [8] for sake of convenience. The condition (ii) is equivalent to $||e^{i(t^{\theta} - I)}|| \leq 1$ for any real $\theta$ and $t \geq 0$, which is in turn equivalent to that $e^{i\theta}T - I$ is dissipative for any real $\theta$, which is identical with $w(T) \leq 1$.

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SECOND PROOF OF THE LEMMA. By the idempotency of $T$ and Theorem A, we have

$$\left\| I + zT + \frac{z^2}{2!} T^2 + \cdots \right\| = \left\| I + (e^z - I)T \right\| \leq e^{|z|}.$$ 

Since $z$ is an arbitrary complex number we put $z = t$, where $t$ is a real number. We get the following inequality

$$\left\| e^{it}I + (1 - e^{-t})T \right\| \leq 1.$$ 

Thus we can conclude $\| T \| \leq 1$ as $t \to \infty$.

Now it is sufficient to show that $T = T^*T$; this method is shown in [4].

$$\|Tx - T^*Tx\|^2 = \|Tx\|^2 - (Tx, T^*Tx) - (T^*Tx, Tx) + \|T^*Tx\|^2$$

$$= \|Tx\|^2 - (T^2x, Tx) - (Tx, T^2x) + \|T^*Tx\|^2$$

$$= \|T^2x\|^2 - \|Tx\|^2 - \|Tx\|^2 + \|T^*Tx\|^2$$

$$= \|T^*Tx\|^2 - \|Tx\|^2 \leq 0;$$

therefore $T$ is a projection. In fact it is well known that an idempotent contraction is a projection.

We shall give a generalization, namely, we can weaken the idempotency of the operator in the Lemma as follows.

**Theorem.** If $T^k = T$ for some integer $k \geq 2$ and if $w(T) \leq 1$, then $T$ is the direct sum of 0 and a unitary operator.

**Proof.** $T^{2(k-1)} = T^{k-2}T = T^{k-2}T = T^{k-1}$. This implies that $T^{k-1}$ is an idempotent operator; also $w(T^{k-1}) \leq (w(T))^{k-1}$ by power inequality [1], [5], [9]. Therefore $T^{k-1} = P$ is a projection by the lemma. Hence we can decompose $T = T_1 \oplus T_2$ where $T_1 = T| R(P)$ and $T_2 = T| N(P)$. Then we have

$$T = T^{k-1}T = (T_1^{k-1} \oplus T_2^{k-1})(T_1 \oplus T_2) = (I \oplus 0)(T_1 \oplus T_2) = T_1 \oplus 0.$$

Now $T_1^{-1} = I$; therefore $w(T_1^{-1}) = w(T_1^{k-2}) \leq (w(T_1))^{k-2} \leq 1$, so that $T_1$ and $T_1^{-1}$ have numerical radius $\leq 1$; hence $T_1$ is unitary [10], so the proof is complete.

**Corollary 1.** If $T^k = T$ and $w(T) \leq 1$, then $T^{k-1}$ is a projection.

This proof is contained in the proof of the Theorem.

We remark that Corollary 1 shows that if $T$ is an idempotent operator that satisfies any of the following conditions:
(i) $T$ is a contraction,  
(ii) $T$ has equal norm and spectral radius (normaloid [5]),  
(iii) $T$ has equal numerical and spectral radius (spectraloid [5]),
then $T$ is a projection.

Moreover we remark that Theorem shows that if $T$ is a periodic operator that satisfies any of the above conditions (i), (ii) and (iii), then $T$ is normal and partial isometric.

**Corollary 2 [4].** If $T$ is paranormal (i.e., if $\|T^2x\| \geq \|Tx\|^2$ for all unit vectors $x$; see [2], [3], [6], [7]) and if $T^k = T$ for some integer $k \geq 2$, then $T$ is normal and partial isometric.

**Proof.** If $T$ is paranormal, then $T$ has norm equal to its spectral radius [2], [3], [7]. Hence if $T^k = T$ for some $k \geq 2$, then $T$ is normal and partial isometric by the above remark.

It is known that this class of paranormal operators properly includes that of hyponormal operators ($T^*T \geq TT^*$) and is properly included in the class of normaloids [2].

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**Added in proof.** A generalization of the results in this paper will appear in the following: T. Furuta, *Some theorems on unitary p-dilations of Sz.-Nagy and Foias* (to appear in Acta Sci. Math.).

**References**


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