ON RINGS WITH A HIGHER DERIVATION

SHIZUKA SATO

Abstract. Let \( R \supset \mathcal{O} \) be two rings with the unit 1. Then we set
\[ \mathfrak{D}(\mathcal{O}, R) = \{ x \in R; x^r \in \mathcal{O} \text{ for some integer } r \geq 1 \}. \]
At first, it is shown that, under some assumptions, \( d \mathcal{O} \subset \mathcal{O} \) implies \( d \mathfrak{D}(\mathcal{O}, R) \subset \mathfrak{D}(\mathcal{O}, R) \). Next, with the Lying-over Theorem on \( d \)-differential ideals, we show: Let \((R, M)\) and \((\mathcal{O}, \mathfrak{m})\) be two quasi-local rings and let \( d \) be a higher derivation of rank 0 of the total quotient ring of \( R \) such that \( d \mathcal{O} \subset \mathcal{O} \). Suppose that \( R \) is integral over \( \mathcal{O} \) and \( \mathcal{O} \) is dominated by \( R \). Then \( d(\mathfrak{m}) \subset \mathfrak{m} \) implies \( d(M) \subset M \).

0. Terminology. In this paper, we assume that all rings are commutative and have the unit 1. Let \( R \supset \mathcal{O} \) be two rings. Then we set:
\[ \mathfrak{D}(\mathcal{O}, R) = \{ x \in R; x^r \in \mathcal{O} \text{ for some integer } r \geq 1 \}, \]
\[ \mathfrak{D}(\mathcal{O}, R)^* = \{ x \in R; \exists y \in \mathcal{O} \text{ such that } yx \in \mathfrak{D}(\mathcal{O}, R) \}. \]

Next, let \( R^{(s)} \) be the set of \( s \)-tuples \((\rho_0, \rho_1, \ldots, \rho_{s-1})\), \( \rho_i \in R \), with operations, for \( x = (x_0, x_1, \ldots, x_{s-1}), y = (y_0, y_1, \ldots, y_{s-1}) \in R^{(s)} \),
\[ x + y = (x_0 + y_0, x_1 + y_1, \ldots, x_{s-1} + y_{s-1}), \]
\[ xy = (z_0, z_1, \ldots, z_{s-1}), \quad \text{where } z_k = \sum_{i+j=k} x_i y_j. \]

This \( R^{(s)} \) is a ring and \( R^{(s)} \) is isomorphic to a formal power series ring \( R[[t]] \) with one indeterminate \( t \) over \( R \).

1. On the existence theorems. Let \( d = (d_i)_{0 \leq i \leq s-1} \) be a higher derivation of rank \( s \) of \( R \) in the sense of P. Ribenboim (cf. [1]). Let \( \mathfrak{A} \) be an ideal of a ring \( R \). Then we shall call \( \mathfrak{A} \) a \( d \)-differential ideal if \( d \mathfrak{A} \subset \mathfrak{A} \), where \( d \mathfrak{A} \subset \mathfrak{A} \) means \( d_i \mathfrak{A} \subset \mathfrak{A} \) for all \( i \).

Theorem 1. Let \( d \) be a higher derivation of rank \( s \) (finite or infinite) of \( R \). If \( \mathfrak{A} \) is a \( d \)-differential ideal of \( R \), then there exists a maximal \( d \)-differential ideal \( \mathfrak{M} \) of \( R \) such that \( \mathfrak{M} \supset \mathfrak{A} \).

Proof. Let \( \mathcal{F} \) be the set of \( d \)-differential ideals containing \( \mathfrak{A} \). Then \( \mathcal{F} \) is an inductive set (the order being given by the inclusion relation). Hence Zorn's Lemma implies the existence of \( \mathfrak{M} \).

The following proposition is almost the same as Theorem 1 in [3]. But the difference between the two propositions is the definition of higher derivations.

Received by the editors November 12, 1970.
AMS 1970 subject classifications. Primary 16A72.
Key words and phrases. Higher derivations, \( d \)-differential ideals, quasi-local rings.

Copyright © 1971, American Mathematical Society

63
Proposition 1. Let \( d = (d_i) \) be a higher derivation of a Noetherian ring \( \mathcal{O} \) such that \( d_0 \) is an isomorphism. Let \( \mathfrak{A} \) be a \( d \)-differential ideal of \( \mathcal{O} \), and let \( \mathfrak{q}_1, \mathfrak{q}_2, \ldots, \mathfrak{q}_r \) be associated prime ideals of \( \mathfrak{A} \). Then \( \mathfrak{q}_1, \mathfrak{q}_2, \ldots, \mathfrak{q}_r \) are \( d \)-differential ideals and \( \mathfrak{A} \) can be written as an irredundant intersection of \( d \)-differential primary ideals.

Proof. We can prove this proposition by almost the same way as Theorem 1 of [3], by considering \( \exp(d), \exp(d') \) instead of \( \exp(tD), \exp(-tD) \) in [3] respectively, where \( d' \) is a higher derivation such that \( dd' = 1 \).

Corollary. With the same \( \mathcal{O}, \mathfrak{A} \) and \( d = (d_i) \) as Proposition 1, there exists always a maximal \( d \)-differential ideal of \( \mathcal{O} \) containing \( \mathfrak{A} \) and this is a prime ideal.

Theorem 2. Let \( R \supseteq \mathcal{O} \) be two rings and let \( d = (d_1, d_2, \ldots, d_s) \) be a higher derivation of the total quotient ring of \( R \) such that \( d \mathcal{O} \subseteq \mathcal{O} \) and \( dR \subseteq R \). Assume that \( R \) is integral over \( \mathcal{O} \). If \( \mathfrak{q} \) is a \( d \)-differential prime ideal of \( \mathcal{O} \), then there exists a \( d \)-differential prime ideal \( \mathfrak{q}' \) of \( R \) such that \( \mathfrak{q}' \cap \mathcal{O} = \mathfrak{q} \).

Proof. First, we shall prove that it is sufficient to consider the case when \( \mathcal{O} \) is an \( d \)-differential quasi-local ring and the maximal ideal of \( \mathcal{O} \) is \( d \)-differential. By the hypothesis, \( d(\mathcal{O}) \subseteq \mathcal{O} \). On the other hand, for \( s \in \mathcal{O} - \mathfrak{q} \), \( d_i(1/s) = f(s)/s^{i+1}, f(s) \in \mathcal{O} \). Since \( d_i \mathcal{O} \subseteq \mathcal{O} \), \( d(\mathfrak{q}_i) \subseteq \mathfrak{q}_i \). Hence \( \mathfrak{q}_i \) is a \( d \)-differential ideal and is the maximal ideal of \( \mathcal{O}_i \). Let \( S = \mathcal{O} - \mathfrak{q} \). Then \( S \) is integral over \( \mathfrak{q} \). Now, suppose that there is a \( d \)-differential prime ideal \( \mathcal{M}' \) of \( R \) such that \( \mathcal{M}' \cap \mathcal{O} = \mathfrak{q} \). Let \( \pi \) be the natural mapping \( \pi(x) = a \cdot 1 \) of \( \mathcal{O} \) into \( \mathcal{O}_i \). Then \( \mathcal{M}' = \pi^{-1}(\mathcal{M}') \) is a \( d \)-differential prime ideal. For let \( x \in \mathfrak{q}' \) and \( \pi(x) = y \in \mathcal{M}' \). Then \( \pi(d_i(x)) = d_i(x) \cdot 1 = d_i(\pi(x)) \) and \( \pi(x) \in \mathcal{M}' \). Hence \( d_i(\pi(x)) \in \mathcal{M}' \) and \( d_i(x) \in \mathfrak{q}' \). Trivially, \( \mathfrak{q}' \cap \mathcal{O} \supseteq \mathfrak{q} \). Conversely, let \( x \in \mathfrak{q}' \cap \mathcal{O} \) and \( \mathcal{M} = \text{Ker}(\pi) \). Further let \( \pi* : \mathcal{O} \rightarrow \mathcal{M} \) be the canonical mapping. \( \mathcal{O}/\mathcal{M} \) is isomorphic to a subring of \( \mathcal{O}_i \). Hence \( \pi*(x) \) is considered as an element of \( \mathcal{O}_i \). Therefore \( \pi*(x) \in \mathcal{M} \cap \mathcal{O}_i \), and \( x \in \mathfrak{q}_i \). So, we have \( x \in \mathfrak{q} \) and \( \mathfrak{q}' \cap \mathcal{O} = \mathfrak{q} \). Thus we may assume that \( \mathcal{O} \) is a quasi-local ring with a higher derivation \( d \) and \( \mathfrak{q} \) is the maximal ideal of \( \mathcal{O} \). It is well known that in this case \( R_\mathfrak{q} \cong R \). Thus, by the Corollary of Proposition 1, there is a maximal \( d \)-differential ideal \( \mathcal{M} \) of \( R \) such that \( \mathcal{M} \supset R_\mathfrak{q} \), and \( \mathcal{M} \cap \mathcal{O} = \mathfrak{q} \).

2. On the invariability concerned with a higher derivation. Let \( d = (d_i) \) be a higher derivation of rank \( s \) (finite or infinite) of \( R \). Then we introduce the ring homomorphism \( \exp(d) \) of \( R^{(s+1)} \) into \( R^{(s+1)} \) as:
for \( x = (x_i) \in R^{(s+1)} \), \( \exp(d)(x) = (z_k) \) where \( z_k = \sum_{i+j=k} d_i(x_j) \).

If \( d_0 \) is an isomorphism, then \( d \) has inverse higher derivation \( \delta \) of rank \( s \) of \( R \), i.e. \( d\delta = 1 \), \( \delta d = 1 \), where \( d\delta = (d_i)(\delta_j) = (\epsilon_k) \), \( \epsilon_k = \sum_{i+j=k} d_i\delta_j \).

Further,

\[
\exp(d\delta) = \exp(d) \exp(\delta) = 1, \quad \exp(\delta d) = \exp(\delta) \exp(d) = 1.
\]

Hence \( \exp(d) \) is an isomorphism [1].

First, we extend a theorem first proved by A. Seidenberg [2, Theorem 1] to the case of a higher derivation in a sense of P. Ribenboim [1].

**Proposition 2.** Let \( R \supseteq \Theta \) be two rings and let \( \Theta' \) be the quasi-integral closure (or the complete integral closure) of \( \Theta \) in \( R \). If \( d \) is a higher derivation of rank \( \omega \) of \( R \) such that \( d\Theta \subseteq \Theta \), then \( d\Theta' \subseteq \Theta' \).

**Proof.** \( R \) is considered as a subring of \( R^{(\omega)} \) by the mapping: \( x \mapsto (x, 0, 0, \ldots) \). Let \( \alpha \) be an element of \( \Theta' \). Then there is an element \( \beta \in \Theta \) such that \( \beta\alpha^p \in \Theta \) for all \( p \geq 0 \). Now, because \( \exp(d)(\Theta) \subset \Theta^{(\omega)} \),

\[
\exp(d)(\beta\alpha^p) = \exp(d)(\beta)\left[\exp(d)(\alpha)^p \in \Theta^{(\omega)} \right] \quad \text{for all } p \geq 0.
\]

Hence \( d_0(\beta)d_0(\alpha)^p \in \Theta \) for all \( p \geq 0 \). On the other hand, \( d_0(\beta) \in \Theta \). Thus, we see that \( d_0(\alpha) \in \Theta' \). Assume that \( d_i(\alpha) \in \Theta' \) for \( i \leq N-1 \). Then,

\[
d_0(\beta)^N \exp(d)(\beta)\left[\exp(d)(\alpha) - (d_0(\alpha), \ldots, d_{N-1}(\alpha), 0, 0, \ldots)\right]^p
\]

\[
= \left(0, \ldots, 0, d_0(\beta)^N d_n(\alpha)^p, \ldots\right) \in \Theta^{(\omega)}.
\]

Therefore, by the induction assumption, \( d_0(\beta)^N d_n(\alpha)^p \in \Theta \) for all \( p \geq 0 \), and \( d_0(\alpha) \in \Theta' \). This completes the proof.

Next, we shall study the relation of \( \Theta(\Theta, R) \) and a higher derivation.

**Theorem 3.** Let \( k \) be a field of characteristic 0 and let \( d = (d_i) \) be a higher derivation of rank \( \omega \) of a domain \( R (\supseteq k) \) such that \( d_0 \) is an isomorphism. Then \( dk \subset k \) implies \( d(\Theta(k, R)) \subset \Theta(k, R) \).

**Proof.** \( d \) can be extended to a higher derivation of \( K (= \text{the quotient field of } R) \). We shall denote by the same \( d \) this extended higher derivation. Let \( 0 \neq x \in \Theta(k, R) \). Then there is an integer \( r \geq 1 \) such that \( x^r \in k \). By the assumption, \( d_i(x^r) \in k \) for all \( i \). Hence, \( \exp(d)(x^r) = [\exp(d)(x)]^r \in k^{(\omega)} \). Thus \( d_0(x)^r \in k \) and \( d_0(x) \in \Theta(k, R) \). Now,
\[ \exp(d)(x^r) - (d_0(x), 0, 0, \ldots)^r \]
\[ = (0, d_1(x), \ldots)(rd_0(x)^{-1}, \ldots) \in k^{(\infty)}. \]
Hence \( rd_0(x)^{-1}d_1(x) \in k \). As \( d_0 \) is an isomorphism and \( d_0(x) \neq 0 \), \( d_0(x)^{-1} \in K \). Further, \( d_0(x)^{r} \in k \) and \( [d_0(x)^{r}]^{-1} \in k \). Thus \( d_0(x)^{-1} = d_0(x)^{-1}[d_0(x)^{r}]^{-1} \in K \). Therefore \( d_0(x)^{-1} \in \Omega(k, R) \). Assume that \( d_i(x) \in \Omega(k, R), i \leq N - 1 \). Then,
\[ \exp(d)(x^r) - (d_0(x), d_1(x), \ldots, d_{N-1}(x), 0, 0, \ldots)^r \]
\[ = (0, \ldots, 0, rd_0(x)^{-1}d_N(x), \ldots) \in \Omega(k, R)^{(\infty)}. \]
Hence \( rd_0(x)^{-1}d_N(x) \in \Omega(k, R) \) and \( d_N(x) \in \Omega(k, R) \). This completes the proof of our assertion \( d(\Omega(k, R)) \subset \Omega(k, R) \).

**Corollary.** Under the assumptions of Theorem 3, \( \Omega(k, R) \) is a field.

**Theorem 4.** Let \( \mathfrak{D} \supset \Omega \) be two domains and let \( d = (d_i) \) be a higher derivation of rank \( \infty \) of \( R \). If \( d_0 \subset \mathfrak{D} \) and \( d(\Omega(0, R)) \subset \Omega(0, R) \), then \( d(\Omega(0, R)^*) \subset \Omega(0, R)^* \).

**Proof.** Let \( x \in \Omega(0, R)^* \). Then there is an element \( y \in \Omega \) such that \( xy \in \Omega(0, R) \). By the hypothesis, \( d_0(yx) = d_0(y)d_0(x) \in \Omega(0, R) \) and \( d_0(y) \in \Omega \). Hence \( d_0(x) \in \Omega(0, R)^* \). Now, \( d_0(y)d_1(yx) = d_0(0)^2d_1(x) + d_0(y)d_0(x)d_1(y), d_0(y)d_1(yx) \in \Omega(0, R) \) and \( d_0(y)d_0(x)d_1(y) \in \Omega(0, R) \). Therefore \( d_0(y)^2d_1(x) \in \Omega(0, R) \) and \( d_1(x) \in \Omega(0, R)^* \). Assume that \( d_i(x) \in \Omega(0, R)^* \) and \( d_i(x)d_0(y)^{i+1} \in \Omega(0, R) \) for all \( i \leq N - 1 \). We have
\[ d_0(y)^N d_N(yx) = d_0(y)^{N+1}d_N(x) + d_0(y)^N \sum_{i+j=N, i \neq 1} d_i(y)d_j(x). \]
Hence \( d_0(y)^{N+1}d_N(x) \in \Omega(0, R) \), by the induction assumption and the fact that \( d_N(x) \in \Omega(0, R)^* \). Thus we have \( d(\Omega(0, R)^*) \subset \Omega(0, R)^* \).

**Corollary 1.** Let \( \mathfrak{D} \supset \Omega \) be two rings and let \( d = (d_i) \) be a higher derivation of rank \( s < \infty \) of \( R \). If \( d_0 \subset \mathfrak{D} \) and \( d(\Omega(0, R)) \subset \Omega(0, R) \), then for any \( x \in \Omega(0, R)^* \), there exists a common element \( y \in \Omega \) such that \( yd_i(x) \in \Omega(0, R) \) for all \( i \).

**Corollary 2.** Under the assumptions of Theorem 3, \( dk \subset k \) implies \( d(\Omega(k, R)) \subset \Omega(k, R) \) and \( d(\Omega(k, R)^*) \subset \Omega(k, R)^* \).

**Proposition 3.** Let \( \mathfrak{D} \supset \Omega \) be two domains and let \( \Omega \) contain the rational number field. Further, let \( d_0 \subset \mathfrak{D} \), then, for any invertible element \( x \) of \( \Omega(0, R) \), \( d_i(x) \in \Omega(0, R) \) for all \( i \).

**Proof.** \( x \in \Omega(0, R) \) implies \( x^r \in \Omega \) for some integer \( r \geq 1 \). By the assumption,
\[ \exp(d)(x^r) = \left[ \exp(d)(x) \right]^r = (d_0(x), d_1(x), \ldots)^r \in \mathfrak{O}^{(\infty)}. \]

Hence \( d_0(x)^r \in \mathfrak{O} \), and \( d_0(x) \in \mathfrak{O}(\mathfrak{O}, R) \). As \( x \) is an invertible element, there is a unique element \( x^{-1} \in \mathfrak{O}(\mathfrak{O}, R) \) such that \( xx^{-1} = 1 \). From the above discussion, \( d_0(x) \in \mathfrak{O}(\mathfrak{O}, R) \). Since \( d_0 \) is a homomorphism, \( d_0(x) \) is an invertible element of \( \mathfrak{O}(\mathfrak{O}, R) \). Now, the \( t \)-th component of \( (d_0(x), d_1(x), \ldots, d_{t-1}(x), \ldots)^r \) is

\[
\begin{pmatrix}
1 \\
\vdots \\
1 \\
\end{pmatrix} d_0(x)^{r-1} d_t(x) + \Delta(d_0(x), d_1(x), \ldots, d_{t-1}(x))
\]

where \( \Delta(d_0(x), d_1(x), \ldots, d_{t-1}(x)) \) is a polynomial in the \( d_0(x), d_1(x), \ldots, d_{t-1}(x) \) with rational coefficients. Assume that, for \( i \leq N - 1 \), \( d_i(x) \in \mathfrak{O}(\mathfrak{O}, R) \). Then, \( (\ldots d_0(x)^{r-1} d_t(x) \in \mathfrak{O}(\mathfrak{O}, R) \). Since \( \mathfrak{O} \) contains rational numbers and \( d_0(x)^{-1} \in \mathfrak{O}(\mathfrak{O}, R) \), \( d_t(x) \in \mathfrak{O}(\mathfrak{O}, R) \). Hence for all \( j \geq 0 \), \( d_j(x) \in \mathfrak{O}(\mathfrak{O}, R) \). This completes the proof.

**Lemma 1.** Assume that a quasi-local ring \( (\mathfrak{O}, \mathfrak{m}) \) is dominated by another quasi-local ring \( (R, M) \). Then \( \mathfrak{O}(\mathfrak{O}, R) \) is a quasi-local ring.

**Proof.** It is sufficient to prove that \( \mathfrak{m} = M \cap \mathfrak{O}(\mathfrak{O}, R) \) is the unique maximal ideal of \( \mathfrak{O}(\mathfrak{O}, R) \). Assume that \( x \in \mathfrak{O}(\mathfrak{O}, R) - \mathfrak{m} \). Then, for some integer \( r \geq 1 \), \( x^r \in \mathfrak{O} \) and \( x \in \mathfrak{m} \). Hence \( x^{-r} \in \mathfrak{O} \) and \( x^{-1} \in R \). Thus \( x^{-1} \in \mathfrak{O}(\mathfrak{O}, R) \).

**Theorem 5.** Let \( (R, M) \) and \( (\mathfrak{O}, \mathfrak{m}) \) be two quasi-local rings and let \( d = (1, d_1, d_2, \ldots) \) be a higher derivation of rank \( \infty \) of the total quotient ring of \( R \) such that \( d \mathfrak{m} \subset \mathfrak{m} \). Suppose that \( R \) is integral over \( \mathfrak{O} \) and \( \mathfrak{O} \) is dominated by \( R \). Then \( d \mathfrak{m} \subset \mathfrak{m} \) implies \( d M \subset M \).

**Proof.** By virtue of Theorem 2, there exists a prime ideal \( M' \) of \( R \) such that \( M' \cap \mathfrak{O} = \mathfrak{m} \) and \( d M' \subset M' \). On the other hand, \( M \supset M' \). Hence \( M \cap \mathfrak{O} \supset M' \cap \mathfrak{O} = \mathfrak{m} \). By the assumption, \( M \cap \mathfrak{O} = \mathfrak{m} \). Hence \( M = M' \). Thus we have \( d M \subset M \).

**Corollary 1.** Let \( (R, M) \) and \( (\mathfrak{O}, \mathfrak{m}) \) be two quasi-local rings and let \( d = (1, d_1, d_2, \ldots) \) be a higher derivation of rank \( \infty \) of the total quotient ring of \( R \) such that \( d \mathfrak{m} \subset \mathfrak{m} \). Assume that \( \mathfrak{O} \) is dominated by \( R \). Then \( d(M \cap \mathfrak{O}(\mathfrak{O}, R)) \subset M \cap \mathfrak{O}(\mathfrak{O}, R) \).

**Proof.** The first half is the consequence of Lemma 1, and the second half is proved the same way as Theorem 5.

**Corollary 2.** Under the assumption of Corollary 1, let \( \mathfrak{O} \) contain the rational number field. Then \( d(\mathfrak{O}(\mathfrak{O}, R)) \subset \mathfrak{O}(\mathfrak{O}, R) \).
Proof. It follows obviously from Proposition 3 and Corollary 1.

Acknowledgement. The author wishes to express his hearty thanks to Professor Y. Nakai for his valuable suggestions.

References


Oita University, Oita, Japan