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THE WORD AND CONJUGACY PROBLEMS FOR THE KNOT GROUP OF ANY TAME, PRIME, ALTERNATING KNOT

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ABSTRACT. The decision problems in the title are solved using the solutions of the word and conjugacy problems obtained by Lyndon and Schupp, respectively, for certain classes of groups.

1. **Introduction.** Each tame knot K has a knot group G whose word problem is solvable by Waldhausen's result [**6**]. Here we give additional results when K is also prime, alternating, and nontrivial. We show that the free product of G and a free group on one generator x_0 is a group H which falls into one of the categories for which Lyndon [**4**] and Schupp [**5**] solved the word and conjugacy problems, respectively. (H has a presentation satisfying the properties C(4) and T_3 [**4**, p. 219].) We thereby obtain another solution of the word problem for G and a solution of the conjugacy problem for G.

In the proof, the given knot is replaced with an equivalent one having a knot-diagram with special properties (referred to as a common knot-diagram). Presentations of G and of another group H are obtained from a knot-diagram, using Dehn's method [**3**, p. 157]. When the knot-diagram is common, the presentation of H (as a factor group of a free group F) satisfies C(4) and T_3. An automorphism of F yields a second presentation of H in which one generator x_0 is not mentioned in the defining relations. Adding the relation $x_0=1$ to this second presentation gives a presentation of G. Thus H is a free product as claimed.

2. **Common knot-diagrams.** Let K be a nontrivial polygonal knot in 3-space R^3. Let K be the range of a continuous function f defined on $[0, 1]$, with $f(0)=f(1)$. Assume K is in regular position with respect to the projection p which sends (x, y, z) to $(x, y, 0)$. (Knot-theoretic terminology is taken from Crowell and Fox [**2**].) The knot-

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diagram $p(K)$ has $m \geqq 3$ double points $V_1, \cdots, V_m$ which are the projections of $2m$ points of $K: f(t_i)$, $1 \leqq i \leqq 2m$, for some t_i in $[0, 1]$. Assume $0 < t_1 < \cdots < t_{2m} = 1$. The *vertices* of $p(K)$ are the V_i and the *edges* are the images, under pf, of $[t_{i-1}, t_i]$, $1 \leqq i \leqq 2m$, where $t_0 = 0$. Observe that each vertex has degree four. If R^2 denotes the plane where $z = 0$, then the complement, in R^2, of $p(K)$ is a disjoint union of $m+2$ *domains* (connected components) $X_0, X_1, \cdots, X_{m+1}$ where X_0 is the unbounded one. We call $p(K)$ *common* if

(i) the boundaries of any two distinct domains have at most one edge in common, and

(ii) each vertex is on the boundary of exactly four distinct domains.

THEOREM A. *Each polygonal, prime, nontrivial knot K is equivalent to a polygonal knot K_0, in regular position, whose knot-diagram $p(K_0)$ is common. If, in addition, $p(K)$ is alternating, then K_0 can be chosen so that $p(K_0)$ is also alternating.*

PROOF. We may assume K is in regular position [**2**, p. 7]. Use induction on the number of vertices of $p(K)$. If either (i) or (ii) fails to hold in $p(K)$, we shall see that K can be replaced by an equivalent polygonal knot K', in regular position, such that $p(K')$ has fewer vertices than $p(K)$, and $p(K')$ is alternating if $p(K)$ is alternating. The proof will be completed by observing that a knot-diagram with at most one vertex corresponds to a trivial knot.

The failure of (i) is treated by an argument of Alexander [**1**, p. 279]. Now suppose (ii) fails at a vertex V_i. The domains which share V_i as a boundary point can be listed in clockwise order as X_a, X_b, X_a, X_c where a, b, c are distinct. V_i is the projection of an overcrossing $f(t_j)$ and undercrossing $f(t_k)$ for some j, k. Assume $t_k = 1$ so that $0 < t_j < 1$. Then there is a simple closed curve C, in R^2, through V_i such that $C - V_i$ lies in X_a and the images, under pf, of $(0, t_j)$ and $(t_j, 1)$ lie in different components of $R^2 - C$.

Let K_1, K_2 be the images, under f, of $[0, t_j]$ and $[t_j, 1]$, respectively. K_1 and K_2, minus their endpoints, lie on different sides of the cylinder $p^{-1}(C)$. Therefore K is the result of joining together in succession the noninterlinking polygonal curves K_1 and K_2.

Since K is nontrivial and prime, one of the curves, say K_1, is knotted and the other one is unknotted. Then K is equivalent to the knot $K'' = K_1 \cup L_1$ where L_1 is the line segment from $f(1)$ to $f(t_j)$. Observe that $p(K'') = p(K_1)$. Finally K'' can be replaced by an equivalent polygonal knot K', in regular position, such that $p(K')$ has fewer vertices than $p(K)$.

3. **The Dehn presentation of** G. Let K be a polygonal knot in regular position. We use the notation from §2, so $p(K)$ has $m+2$ domains. Let F be a free group on $m+2$ generators $x_0, x_1, \cdots, x_{m+1}$. (The generator x_j corresponds to a path, starting at a distant base point on the z-axis, crossing the domain X_i, passing below the plane where $z=0$, and returning to the base point by way of X_0.) Each vertex V_i of $p(K)$ determines one defining relator r_i as follows.

The domains which share V_i as a boundary point can be listed, possibly redundantly, in clockwise order X_a, X_b, X_c, X_d where a, b, c, d are not necessarily distinct. Observe that they are distinct when $p(K)$ is common. V_i is the projection of some overcrossing $f(t_j)$ and some undercrossing $f(t_k)$. In K, we choose two arcs, an overpass O containing $f(t_j)$ and an underpass U containing $f(t_k)$. Let K be given an orientation, with O, U oriented accordingly. The subscripts in the above listing are to be chosen so that X_a meets X_b along an initial part of O. Put $r_i = x_a x_b^{-1} x_c x_d^{-1}$, which is a cyclically reduced word. Then the group of the knot K is

$$G = \langle x_0, x_1, \cdots, x_{m+1}; r_1, \cdots, r_m, x_0 \rangle.$$

The result of removing all traces of x_0 from this presentation can be described using the endomorphism E of F determined by $E(x_0)=1$ and $E(x_j)=x_j$, $1 \leqq j \leqq m+1$.

REMARK. $G \cong \langle x_1, \cdots, x_{m+1}; E(r_1), \cdots, E(r_m) \rangle$.

The promised group H is given by

$$H = \langle x_0, x_1, \cdots, x_{m+1}; r_1, \cdots, r_m \rangle.$$

LEMMA. *H is the free product of G and the free group on one generator x_0.*

PROOF. Let A be the automorphism of F determined by $A(x_0)=x_0$ and $A(x_j)=x_j x_0$, $1 \leqq j \leqq m+1$. With E as before, some checking shows that the cyclically reduced form of $A(r_i)$ equals $E(r_i)$, $1 \leqq i \leqq m$. Thus

$$H \cong \langle x_0, x_1, \cdots, x_{m+1}; A(r_1), \cdots, A(r_m) \rangle$$

and, by the remark,

$$G \cong \langle x_1, \cdots, x_{m+1}; A(r_1), \cdots, A(r_m) \rangle$$

where the cyclically reduced form of each $A(r_i)$ does not involve x_0. This proves the lemma.

When $p(K)$ is common, the presentation of H has the property C(4) which is defined and verified in Theorem B. The definition of

C(4) requires Lyndon's concept of a *piece* (relative to a subset R of F) which is an element u of F such that R contains two distinct elements with reduced forms $r=uv$ and $s=uw$.

THEOREM B. *Let K be a polygonal, prime knot, in regular position, with a common knot-diagram $p(K)$ which is alternating. Let $r_1, \cdots, r_m$ be the defining relators, of length 4, in a Dehn presentation of the knot group of K. Let R be the set consisting of r_i, r_i^{-1}, $1 \leqq i \leqq m$, and all cyclically reduced conjugates of these elements. Then R satisfies:*

C(4): *No member of R is a product of fewer than 4 pieces, and*

T_3: *if r, s, t are in R, then in at least one of the products rs, st, tr there is no cancellation.*

PROOF. If C(4) fails, then there is some piece $x_jx_k^{-1}$ (or $x_j^{-1}x_k$) coming from two distinct elements r, s in R. The boundaries of the corresponding domains X_j, X_k have exactly one edge in common, since $p(K)$ is common. This edge is the image, under pf, of $[t_{i-1}, t_i]$, for some i, using the notation from §2. Then r, s correspond to vertices $pf(t_{i-1})$ and $pf(t_i)$ where $f(t_{i-1})$ and $f(t_i)$ are two successive overcrossings (or undercrossings). This is impossible since $p(K)$ is alternating. Thus C(4) holds.

If T_3 fails, there exist domains X_i, X_j, X_k such that X_i, X_j have a common boundary edge E_{ij} and the pairs X_j, X_k and X_k, X_i have the same property, for edges E_{jk}, E_{ki}, respectively. Let P denote the planar graph determined by the vertices and edges of $p(K)$. Consider the graph P^* which is dual to P. The faces X_i, X_j, X_k and the edges E_{ij}, E_{jk}, E_{ki} of P correspond to parts of a triangle in P^*, namely, to vertices X_i^*, X_j^*, X_k^* and edges E_{ij}^*, E_{jk}^*, E_{ki}^*, respectively. In P^*, there is a subgraph S^* consisting of this triangle and all vertices and edges interior to it. Observe that each face of S^* has four boundary edges since each vertex of P has degree four. If e^*, f^* denote the numbers of edges and faces, respectively, of S^*, we find $2e^*=3+4f^*$, an impossible equation. Thus, T_3 holds, and we are done.

4. Main result.

THEOREM C. *Let K be a tame, prime knot, in regular position, with a knot-diagram $p(K)$ which is alternating. Let G be the knot group of K. Then the word and conjugacy problems for G are solvable with respect to a Dehn presentation of G.*

PROOF. We may assume that K is polygonal, nontrivial, and, by Theorem A, that $p(K)$ is common. By Theorem B, Lyndon's [4] Theorem II, and Schupp's [5] Theorem 4.1, the word and conjugacy problems are solvable for H. Hence, by the lemma, the same is true for G.

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