PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 30, No. 1, September 1971

CONCERNING UPPER SEMICONTINUOUS DECOMPOSITIONS OF IRREDUCIBLE CONTINUA

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ABSTRACT. Let $\mathcal{K}$ denote the class of all compact metric continua K such that there exists a monotone mapping from a compact metric irreducible continuum M onto an arc such that each point inverse is homeomorphic to K. It is shown that no connected 1-polyhedron other than an arc is an element of $\mathcal{K}$, but that $\mathcal{K}$ contains certain locally connected continua.

Let $\mathcal{K}$ denote the class of all compact metric continua K such that there exists an upper semicontinuous decomposition G of a compact metric irreducible continuum M with each element of G homeomorphic to K and the decomposition space M/G an arc, or, equivalently, that there exists a monotone mapping from a compact metric irreducible continuum M onto an arc such that each point inverse is homeomorphic to K. In 1935, B. Knaster [**2**] showed that an arc is in $\mathcal{K}$. It is known that every compact metric continuum can be embedded in a member of $\mathcal{K}$. Recently, W. S. Mahavier [**4**] has shown that if K is any compact metric continuum there is a monotone open mapping from a compact irreducible Hausdorff continuum M onto an arc such that each point inverse is homeomorphic to K. He raises the question of whether a simple closed curve is in $\mathcal{K}$. Theorem 1 answers this in the negative. Theorem 2 shows that there are locally connected members of $\mathcal{K}$ which are not themselves irreducible. Theorem 3 shows that the arc is the only connected finite 1-polyhedron in $\mathcal{K}$. The theorems are given in the order in which they were obtained.

THEOREM 1 (TRANSUE). *No simple closed curve is in* $\mathcal{K}$.

PROOF. Assume the contrary. Let G be an upper semicontinuous decomposition of the irreducible continuum M such that the elements of G are simple closed curves and M/G is an arc, and let f be a resultant monotone map of M onto $[0, 1]$ whose point inverses are the elements of G. Now G is continuous except on a set of the first category [**3**, p. 48], so if u is an open subset of $[0, 1]$ then there are two points a and b of u, such that $[a, b]\subset u$ and G is continuous at $f^{-1}(a)$ and $f^{-1}(b)$. It follows that $f^{-1}([a, b])$ is irreducible from $f^{-1}(a)$ to

Received by the editors October 16, 1970.

AMS 1969 *subject classifications*. Primary 5455; Secondary 5460.

Key words and phrases. Upper semicontinuous decomposition, irreducible metric continua, 1-polyhedra, simple closed curve.

$f^{-1}(b)$. E. Dyer [1] has shown that there is no continuous collection of decomposable continua, the union of whose elements is an irreducible continuum. Therefore, G must be discontinuous at each element of a dense set.

There exist sequences t, h, and V such that:

(1) G is discontinuous at $f^{-1}(t_i)$, $i=1, 2, \cdots$;

(2) h_i is a proper subcontinuum of $f^{-1}(t_i)$, therefore chainable, which is the limit (in the Hausdorff metric) of some sequence of elements of G;

(3) V_i is the union of the elements of some $1/i$-chain of open sets covering h_i;

(4) $f^{-1}(t_{i+1}) \subset V_i$; and

(5) $\overline{V}_{i+1} \subset V_i \cap f^{-1}((t_{i+1}-(i+1)^{-1}, t_{i+1}+(i+1)^{-1}))$.

Then $\bigcap_{i=1}^{\infty} V_i = f^{-1}(\bigcap_{i=1}^{\infty} f(V_i))$ is an element of G which is chainable, a contradiction.

THEOREM 2 (FITZPATRICK). *There exists a compact locally connected continuum, not an arc, which is a member of* $\mathcal{K}$.

PROOF. Consider, in E^3, the plane:

$$\{(x, y, 0) \mid (x, y) \in E^2\}.$$

There exists, in this plane, a dendron (a locally connected continuum containing no simple closed curve) K with a countable dense subset $C = \{c_1, c_2, c_3, \cdots\}$ such that if $p = (p_1, p_2, 0)$ is a point of C, then

$$K \cup \{(x, y, 1) \mid (x, y, 0) \in K\} \cup \{(p_1, p_2, t) \mid 0 \leqq t \leqq 1\}$$

is homeomorphic to K. Let T denote the Cantor middle-third set on $[0, 1]$. The elements of G are defined as follows. If $t \in T$ is not an endpoint of a component of $[0, 1] \backslash T$, then $\{(x, y, t) \mid (x, y, 0) \in K\}$ is an element of G. If r and s are the endpoints of a component of $[0, 1] \backslash T$ of length 3^{-n}, and $c_n = (p_1, p_2, 0)$ then

$$\{(x, y, r) \mid (x, y, 0) \in K\} \cup \{(x, y, s) \mid (x, y, 0) \in K\} \cup \{(p_1, p_2, t) \mid r \leqq t \leqq s\}$$

is an element of G.

Let M be the union of all the elements of G. Clearly G is upper semicontinuous, M/G is an arc, and each element of G is homeomorphic to K. To see that M is irreducible note that each point (x, y, t), where $(x, y, 0) \in K$ and $t \in T$, is a limit point of the union of the vertical intervals added in.

REMARKS. In trying to obtain such an example where no element of

G has interior in M, one might simply take the M above, with each vertical interval collapsed to a point. But then some elements of G have junction points of Menger order 4 [**5**, p. 292] and others do not. This difficulty can be avoided by altering K so that all of its junction points are of Menger order 4; this can be done in such a way that K contains a countable dense subset C such that if $p=(p_1, p_2, 0)$ belongs to C then the upper semicontinuous decomposition of $K \cup \{(x, y, 1) \mid (x, y, 0) \in K\} \cup \{(p_1, p_2, t) \mid 0 \leqq t \leqq 1\}$ whose only nondegenerate element is $\{(p_1, p_2, t) \mid 0 \leqq t \leqq 1\}$ is homeomorphic to K.

It should also be clear that a K as in Theorem 2 may be taken to contain simple closed curves. In fact, the following may be proved by an argument similar to the preceding, except that the construction may have to take place in a higher dimensional embedding space.

THEOREM 2'. *Suppose that K is a compact metric continuum and that there exists a countable collection $\mathcal{C}$ of closed subset(s) of K such that no proper subcontinuum of K contains all the elements of $\mathcal{C}$ and such that if $C \in \mathcal{C}$ and K' is a copy of K not intersecting K (except perhaps at C) and C' is the induced copy of C in K', then there is an irreducible continuum I from K to K' such that $K \cap I = C$, $K' \cap I = C'$, and $I \cup K \cup K'$ is homeomorphic to K. Then K is a member of $\mathcal{K}$.*

THEOREM 3 (HINRICHSEN). *The arc is the only connected finite 1-polyhedron which is a member of $\mathcal{K}$.*

PROOF. Assume the contrary. Let Y be a connected finite 1-polyhedron, not an arc, and let G be an upper semicontinuous decomposition of the irreducible continuum M such that the elements of G are homeomorphic to Y and such that M/G is an arc. Let ρ be a metric on M.

By Theorem 1, Y is not a simple closed curve. Therefore, Y contains a junction point, i.e., a point of Menger order greater than two. For each element g of G let F_g denote the set of all junction points of g. If F_g is degenerate, let δ_g denote a positive number such that $\rho(F_g, y) > \delta_g$ for some y in g. If F_g is nondegenerate let δ_g denote a positive number such that if x and y are two points of F_g then $\rho(x, y) > 2\delta_g$. Let D_g denote the collection of spherical neighborhoods (in g) of points of F_g of radius δ_g; the closures of the elements of D_g are mutually exclusive. If g is in G there exists a finite point set E_g such that $g \backslash E_g$ is the union of a finite number of mutually separated sets each containing only one element of D_g.

If g is in G and P is a point of F_g denote by $t(P)$ a simple triod with P as its junction point and lying in the element of D_g which contains

P, and let $T_g = \{t(P) \mid P \in F_g\}$. If x is a nonjunction point of a simple triod t let t_x denote the arc of t containing two endpoints of t but not x. If $x \in g \in G$, $x \notin F_g$, let x_g denote the component of $g \setminus F_g$ containing x. $\bar{x}_g$ is an arc or a simple closed curve.

If each of c and d is a positive integer let H_{cd} denote the collection of all elements h of G such that:

(1) $1/c < \delta_h$;

(2) $\rho(E_h, F_h) > 2/c$;

(3) if $P \in F_h$ and x is an endpoint of $t(P)$, then $\rho(x, t(P)_x) > 2/c$;

(4) if $P \in F_h$ and $x \in t(P)$ and $\rho(x, P) > 1/8c$, then $\rho(x, t(P)_x) > 1/d$; and

(5) if $x \in h$ and $\rho(x, F_h) > 1/8c$, then $\rho(x, h - x_h) > 1/d$.

Note that every element of G belongs to some H_{cd}.

Suppose α is a convergent (in the Hausdorff metric) sequence of simple triods such that if z is a term of α, there is an element h of H_{cd} such that $z \in T_h$. The limit L of α is a nondegenerate subcontinuum of a connected 1-polyhedron, so it is a connected 1-polyhedron. Now L is not an arc, for suppose it is. Denote by V a linear chain of open sets covering L each of diameter less than $1/c$. Some term t of α is a subset of V^* (the union of all the elements of V). But some endpoint x of t is contained in an element of V which intersects t_x, contrary to condition (3) above. A similar argument shows that L is not a simple closed curve. Therefore L contains a junction point.

Now consider $\overline{H}_{cd}$, the closure being taken in M/G. If each $\overline{H}_{cd}$ were totally disconnected then by [5, p. 23, Theorem 57], M/G would not be a continuum. Therefore some $\overline{H}_{cd}$ contains an arc H of elements of G. Let $H' = H_{cd} \cap H$, $F_{H'} = \bigcup_{h \in H'} F_h$. Let u denote a cut (i.e., separating) element of H that belongs to H'.

Suppose there exists a sequence α of elements of H' which converges to a subset of u containing a point x such that $\rho(x, F_u) \geqq 1/2c$ and x is a limit point of the set of junction points of terms of α. Let C denote a polyhedral chain of open sets in Y irreducibly covering u such that: each element of C intersects u in a connected set; the diameter of each element of C is less than $1/4d$ and $1/16c$; there is a subcollection C' of C irreducibly covering F_u such that no two elements of C' intersect and such that $C \setminus C'$ is the union of a finite collection of mutually exclusive linear chains.

Let z denote the component of $u \cap N_{1/4c}(x)$ that contains x. $\bar{z}$ is an arc, since it is a proper subset of x_u. $\bar{z}$ has at least one but no more than two boundary points with respect to u. Suppose $\bar{z}$ has two boundary points with respect to u, q_1 and q_2. Since $\rho(\bar{z}, F_u) > 1/8c$ then $\rho(\bar{z}, u/x_u) > 1/d$. Assume there is an element v of C which contains q_1

and q_2. Since $v\cap u$ is connected, there is an arc of u from q_1 to q_2 lying in v; this arc must be $\bar{z}$. Therefore, $\bar{z}$ is a subset of v. But the diameter of $\bar{z}$ is at least $1/4c$, and the diameter of v is less than $1/8c$, an impossibility. Let v_1 and v_2 denote two elements of C which contain q_1 and q_2, respectively. Since $\rho(\bar{z}, u\backslash x_u) > 1/d$, the union of the linear subchain C_{v_1,v_2} of C from v_1 to v_2 that contains x does not intersect $u\backslash x_u$. There exists a term k of α and a point f of F_k such that $k\subset C^*$ and $\rho(f, x) < 1/16c$. Since each endpoint of $t(f)$ is at a distance greater than $2/c$ from f and diameter $C_{v_1,v_2} < 1/c$, then two arcs of $t(f)$ intersect one of v_1 and v_2. But then there is a point P of $t(f)$ such that $\rho(P, f) > 1/8c$ and $\rho(P, t(f)_P) < 1/4d$, contrary to condition (4) above. A similar argument holds in case $\bar{z}$ has only one boundary point with respect to u.

Therefore, there exists an arc K of elements of H such that u is a cut element of K and $F_{H'}\cap K^*\subset N_{1/2c}(F_u)$. Let D denote the collection of all $1/2c$-neighborhoods of points of E_u, and D' the collection of all $1/4c$-neighborhoods of points of E_u. Let A and B denote the end elements of K. Let R, respectively R', denote the collection of all $1/c$-neighborhoods, respectively, $1/2c$-neighborhoods of points of F_u.

If $k\in H'\cap K$ and $q'\in R'$, one and only one point of F_k belongs to q'; and hence, if k is in K and q is in R, one and only one point of F_k belongs to q. Let f denote a point of F_u and q_f and q_f' denote the elements of R and R', respectively, that contain f. The elements of $R\cup D$ are mutually exclusive. Let $J = D\cup(R\backslash q_f)$ and $J' = D'\cup(R'\backslash q_f')$. No point of $F_H\cap K^*\cap q_f$ is in $\bar{J}^*$.

Now let V' denote the collection to which v belongs if and only if: (1) v is the component of the intersection of $K^*\backslash J'^*$ with an element h of $H'\cap K$, and (2) v intersects F_h. Each element h of $H'\cap K$ contains one and only one element of V' and that element of V' contains one and only one point of F_h. If v is an element of V' and h is the element of $H'\cap K$ that contains v, then

(i) v contains one and only one point P of F_h;

(ii) the component of $v\cap t(P)$ that contains P is a triod whose junction point is not in J^*;

(iii) each endpoint of $v\cap t(P)$ is at a distance at least $1/4c$ from the junction point of v; and hence

(iv) if x is an endpoint of $v\cap t(P)$ then $\rho(x, (t(P)\cap v)_x) > 1/d$.

Hence, as before, the sequential limit of a convergent sequence of elements of V' contains a triod. Therefore, if $g\in k$, only one component of $g\cap(K^* - J'^*)$ contains a simple triod and this component contains only one point of F_g.

Let V denote the collection to which v belongs if and only if v is in V' or v is a component of the intersection of an element of K with

$K^*\backslash J'^*$ that contains the sequential limit of a convergent sequence of elements of V'. Then V^* is closed and intersects each element of K and is a subset of $K^*\backslash J'^*$.

Since M is irreducible, $A\cup B\cup(K^*\backslash J'^*)$ is the sum of two mutually exclusive closed point sets, M_A and M_B, containing A and B, respectively [5, p. 15, Theorem 44]. Let V_A and V_B denote the set of all elements of V in M_A and M_B, respectively. Let v_1 denote the last (in the arc V in the order from A to B) element of V_A, and v_2 denote the first element of V_B which follows v_1. Let g_1 and g_2 denote the elements of G which contain v_1 and v_2, respectively. $g_1 \neq g_2$, since no element of G contains two elements of V. There is an element g' of G between g_1 and g_2; g' contains an element of V between v_1 and v_2. This involves a contradiction.

THEOREM 3'. *Suppose K is a compact metric continuum and there exists a compact metric continuum T which is irreducible from K to a copy K' of K and such that $K\cup T\cup K'$ is homeomorphic to K. Then K belongs to $\mathcal{K}$.*

PROOF. Suppose I^∞ is the Hilbert cube, $I^\infty=[0, 1]\times[0, 1]\times\cdots$. We may suppose K is a continuum in I^∞. Let C be the Cantor middle-third set in $[0, 1]$; for each α in C,

$$K_\alpha = \{(x_1, x_2, x_3, \cdots) \mid x_i \in [0, 1],\ x_1 = \alpha,\ (x_2, x_3, \cdots) \in K\};$$

$s_1, s_2, s_3, \cdots$ are the components of $[0, 1]\backslash C$, $s_i=\{t \mid a_i<t<b_i\}$. Then there exists a sequence of copies of T: $T_1, T_2, T_3, \cdots$ each lying in I^∞, such that

(1) T_j is an irreducible continuum from K_{a_j} to K_{b_j},

(2) $K_{a_j}\cup K_{b_j}\cup T_j$ is homeomorphic to K and

(3) the boundary in I^∞ of $\bigcup_{j=1}^\infty [T_j\backslash(K_{a_j}\cup K_{b_j})]$ is $\bigcup_{\alpha\in C}K_\alpha$.

The dendron K in Theorem 2 has uncountably many endpoints. Using Theorem 3' one can obtain a dendron K in $\mathcal{K}$ with only countably many endpoints; let $t_1, t_2, t_3, \cdots$ denote the endpoints of C = Cantor set in $[0, 1]$ and let

$$K = \{[0, 1] \times \{0\}\} \cup \bigcup_{i=1}^{\infty} \{\{t_i\} \times [0, i^{-1}]\}.$$

One may still wonder whether a 2-cell is in $\mathcal{K}$, or, indeed, whether $\mathcal{K}$ contains every locally connected continuum which does contain two disjoint copies of itself, and whether $\mathcal{K}$ contains any locally connected continuum that does not.

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