TAME POLYHEDRA IN WILD CELLS AND SPHERES

R. B. SHER

Abstract. It is shown that each arc on a disk $D$ in $E^4$ can be homeomorphically approximated by an arc in $D$ which is tame in $E^4$. Some applications of this are given. Also, we construct an everywhere wild $(n-1)$-sphere in $E^n$, $n \geq 3$, each of whose arcs is tame in $E^n$.

1. Introduction. In [11] Seebeck showed that each arc on a disk $D^2 \subset E^n$, $n \geq 5$, can be homeomorphically approximated by arcs in $D^2$ which are tame in $E^n$. He also showed that if $n \geq 4$ and $2 \leq m \leq n-2$, then there is an $m$-cell in $E^n$ which is wild at each of its points but each of whose arcs is tame in $E^n$.

In the current paper, we use some recent work of Bryant [3] to eliminate the missing case $n = 4$ in the first result, and to sharpen the second result to the case $n \geq 4$ and $2 \leq m \leq n-1$. We also give applications of the first result to problems in disk squeezing and cellularity.

We use $E^n$ to denote Euclidean $n$-space and $d$ to denote its usual metric. We sometimes regard $E^n$ as being the $(x_{n+1} = 0)$-hyperplane in $E^{n+1}$. In this way, if $X \subset E^n$ is compact we may regard the suspension of $X$, denoted $\Sigma X$, as a subset of $E^{n+1}$. It is the join of $X$ with the set $\{(0, \cdots, 0, -1), (0, \cdots, 0, 1)\} \subset E^{n+1}$.

If $M$ is a manifold we use $\text{Int} M$ and $\partial M$ for the interior and boundary of $M$ respectively. We use $\Delta^n$ to denote the standard $n$-simplex and $S^{n-1}$ to denote $\partial \Delta^n$.

2. Approximating 1-complexes by tame ones. Our main tool will be the following result due to Bryant [3, Corollary 2]. For completeness, we state it here. For convenience, we have modified its form slightly.

Theorem 1 (Bryant). Let $A$ be an arc in $E^{n+1}$, $n \geq 3$. Suppose there exists a dense subset $\{t_1, t_2, \cdots\}$ of $R$ such that $(E^n \times \{t_i\}) - A$ is 1-ULC for each $i$. Then $A$ is tame.

Received by the editors October 2, 1970.

AMS 1970 subject classifications. Primary 57A15, 57A45; Secondary 55A30, 57A60.

Key words and phrases. Everywhere wild, homeomorphically approximate, disk squeezing, cellularity.

This research was supported in part by National Science Foundation Grant GP-19961.
Having set forth the tool, we may now state the theorem whose proof occupies this section.

**Theorem 2.** Let $K$ be a finite 1-complex, $h : K \to \Delta^2$ an embedding, $f : \Delta^2 \to E^4$ an embedding, and $\epsilon > 0$. Then there is an embedding $g : K \to \Delta^2$ such that $fg : K \to E^4$ is tame and $d(fg(x), fh(x)) < \epsilon$ for all $x \in K$. Furthermore, if $L$ is a subcomplex of $K$ such that $fh|L$ is tame, then $g$ can be chosen so that $g|L = h|L$.

**Proof.** The method of proof is inductive, so we may suppose that $K - L$ consists of a single 1-simplex, say $\sigma$, with vertices $a$ and $b$. We shall construct an $\epsilon/4$-push $k$ of $(E^4, f(\Delta^2))$ and an embedding $g_* : \sigma \to \Delta^2$ such that

1. $g_*\{a, b\} = h\{a, b\}$,
2. $g_*|h(K - \sigma) = \emptyset$,
3. $d(kfg_*(x), kfh(x)) < \epsilon/2$ for all $x \in \sigma$, and
4. $kfg_*$ is tame.

Then $g$ is given by $g|K - \sigma = h|K - \sigma$ and $g|\sigma = g_*$. By our assumption, $fg$ is tame on $L$ and, since $k$ is a push, $fg$ is tame on $\sigma$. Hence, $fg$ is tame by [4, Added in proof]. If $x \in \sigma$, then

$$d(fg(x), fh(x)) = d(k^{-1}[kfg_*(x)], k^{-1}[kfh(x)])$$

$$\leq d(k^{-1}[kfg_*(x)], kfg_*(x)) + d(kfg_*(x), kfh(x)) + d(kfh(x), k^{-1}[kfh(x)])$$

$$< \epsilon/4 + \epsilon/2 + \epsilon/4 = \epsilon.$$

To construct $k$ and $g_*$ we begin by selecting a countable dense subset $\{t_1, t_2, t_3, \ldots\}$ of $R$. We may suppose that $\{fh(a), fh(b)\} \subset E^4 - \bigcup_{i=1}^\infty E^3 \times \{t_i\}$. Let $Q_i$ be a subpolyhedron of $E^4$ triangulating $E^3 \times \{t_i\}$, let $Q'_i$ denote the $r$th barycentric subdivision of $Q_i$ (with $Q_0 = Q$), and let $T'_i$ denote the 2-skeleton of $Q'_i$. Let $\epsilon_i = \epsilon/8$. By [11, Lemma 2], there is an $\epsilon_i$-push $k_i$ of $(E^4, f(\Delta^2) \cap T'_{i-1})$ such that $k_i(f(\Delta^2) \cap T'_{i-1})$ is at most 0-dimensional. Let $D^2$ be a disk in $\Delta^2$ such that $h(\sigma)$ spans $D^2$ and $\partial D^2 \cap h(K) = \{h(a), h(b)\}$. There is a proper embedding $g_i : \sigma \to D^2$ such that $g_i|\{a, b\} = h|\{a, b\}$, $kfg_i(\sigma) \cap T^0 = \emptyset$, and $d(kfg_i(x), kfh(x)) < \epsilon_i$ for all $x \in \sigma$.

Inductively, suppose we have constructed, $\epsilon_i$, $k_i$, and $g_i$. Let

$$\delta_i = \min\{d(kfg_i(\sigma), T_1 \cup T_2 \cup \cdots \cup T_i), \delta_1, \delta_2, \ldots, \delta_{i-1}\}$$

and let

$$\epsilon_{i+1} = \min\{\epsilon_i/2, \delta_i/2^{i+1}\}.$$

By [11, Lemma 2], there is an $\epsilon_{i+1}$-push $k_{i+1}'$ of $(E^4, k_i(f(\Delta^2) \cap (T_i \cup$
... \cup T^1_i \cup T^1_{i+1})$ such that $k_{i+1} \cdot k_i(f(D^2)) \cap (T^1_i \cup T^1_{i-1} \cup ... \cup T^1_i \cup T^1_{i+1})$ is at most 0-dimensional. Let $k_{i+1} = k_i \cdot k_i$ and let $g_{i+1} : \sigma \to D^2$ be a proper embedding such that

$$g_{i+1} \vert \{a, b\} = h \vert \{a, b\},$$

and

$$k_{i+1} \cdot g_{i+1}(\sigma) \cap (T^1_i \cup T^1_{i-1} \cup ... \cup T^1_i \cup T^1_{i+1}) = \emptyset,$$

and

$$d(k_{i+1} \cdot g_{i+1}(x), k_{i+1} \cdot g_{i}(x)) < \epsilon_{i+1} \text{ for all } x \in \sigma.$$

We may also pick $\epsilon_{i+1}$ small enough (depending on $k_i$ and $g_i$) so that $\{k_i\}_{i=1}^\infty$ converges to a push and $\{g_i\}_{i=1}^\infty$ converges to an embedding [7].

Let $k = \lim_{i \to \infty} k_i$ and $g = \lim_{i \to \infty} g_i$. Then $k$ is an $\epsilon/4$-push of $(E^4, f(D^2))$. Property (1) holds since $g_i \vert \{a, b\} = h \vert \{a, b\}$ for all $i$, and property (2) holds since $g_i$ embeds $\sigma$ in $D^2$. To see that property (3) holds, let $x \in \sigma$. Then

$$d(k \cdot g_i(x), k \cdot h(x)) \leq d(k \cdot g_i(x), k \cdot h(x))$$

$$+ \sum_{i=1}^\infty d(k \cdot g_i(x), k_{i+1} \cdot g_{i+1}(x)) + \sum_{i=1}^\infty d(k_{i+1} \cdot g_{i+1}(x), k_{i+1} \cdot g_{i}(x)) < \epsilon_1$$

$$+ \sum_{i=1}^\infty \epsilon_i + \sum_{i=1}^\infty \epsilon_{i+1} = 2 \sum_{i=1}^\infty \epsilon_i \leq \epsilon/2.$$

We complete the proof by showing that $k \cdot g_i$ is tame. Let $x \in \sigma$ and $j, k$ be positive integers. Then

$$d(k \cdot g_i(x), T^{k}_{j}) \leq d(k_{j+k} \cdot g_{j+k}(x), T^{k}_{j})$$

$$\leq \sum_{i=1}^\infty d(k \cdot g_i(x), k_{i+1} \cdot g_{i+1}(x)) \leq \delta_{j+k} - 2 \sum_{i=1}^\infty \epsilon_i$$

$$\leq \delta_{j+k} - 2 \sum_{i=1}^\infty \delta_i/2^{i+1} \leq \delta_{j+k} - 2 \sum_{i=1}^\infty \delta_{j+k}/2^{i+1}$$

$$= \delta_{j+k} - 2(\delta_{j+k}/2^{j+k}) > 0.$$
3. Disk squeezing and cellularity. The theorem stated below was proved by R. J. Daverman and W. T. Eaton for the case \( n = 3 \) [5] and \( n \geq 5 \) [6]. Theorem 2 (in place of [6, Lemma A]) allows the proof of [6, Theorem 2] to go through in the case \( n = 4 \).

The notation of the theorem is as follows: \( \Delta_3 \) is the disk in \( \mathbb{E}^3 \) given by \( \{ (x, y) \mid x^2 + y^2 \leq 1 \} \), \( \Delta_1 \) is the arc spanning \( \Delta_3 \) and given by \( \{ (x, 0) \mid -1 \leq x \leq 1 \} \), and \( \pi \) is the projection of \( \Delta_3 \) onto \( \Delta_1 \) given by \( \pi(x, y) = (x, 0) \).

**Theorem 3** (Daverman and Eaton). Suppose \( D^2 \) is a disk in \( \mathbb{E}^n \), \( n \geq 3 \), \( g_0 : \Delta_3 \to D^2 \) is a homeomorphism, \( U \) is an open subset of \( \mathbb{E}^n \) with \( D^2 - g_0(\partial \Delta_3) \subseteq U \) and \( \epsilon > 0 \). Then there exists a surjective map \( f : E^n \to E^n \), a homeomorphism \( g : \Delta_3 \to D^2 \), and a homeomorphism \( \alpha : \Delta_1 \to f(D^2) \) such that

1. \( f| E^n - D^2 \to E^n - f(D^2) \) is a homeomorphism,
2. \( f| E^n - U = \text{id}_{E^n - U} \),
3. \( fg = h \pi \),
4. \( d(g(x), g_0(x)) < \epsilon \) for all \( x \in \Delta_3 \), and
5. \( g| \partial \Delta_3 = g_0| \partial \Delta_3 \).

**Remark.** (iii) It is interesting to note that, in case \( n = 4 \), the arcs \( f^{-1}(p), p \in \text{Int} f(D^2) \), are cellular in \( E^n \) (definition below). This follows from [1, Corollary 5.5].

Applying Theorem 3 to a wedge of disks in \( E^n \) we obtain the following result.

**Corollary 4** (Daverman and Eaton). Suppose that \( P^2 = (D^2_1, a) \lor (D^2_2, a) \lor \cdots \lor (D^2_k, a) \subseteq \mathbb{E}^n \), where \( a \in \partial D^2_i \) if \( i = 1, 2, \ldots, k \), and that \( U \) is an open subset of \( \mathbb{E}^n \) with \( P^2 \subseteq U \). Then there exists a surjective map \( f : \mathbb{E}^n \to \mathbb{E}^n \), homeomorphisms \( g_i : \Delta_3 \to D^2_i \) and homeomorphisms \( h_i : \Delta_1 \to f(D^2_i) \) such that

1. \( f| E^n - P^2 \to E^n - f(P^2) \) is a homeomorphism,
2. \( f| E^n - U = \text{id}_{E^n - U} \),
3. \( g_i = h_i \pi \) for \( i = 1, 2, \ldots, k \), and
4. \( g_i((-1, 0)) = a = f(a) \) for \( i = 1, 2, \ldots, k \).

Recall that a continuum \( X \) in the interior of an \( n \)-manifold \( \mathbb{M}^n \) is **cellular** if there exist \( n \)-cells \( C_1, C_2, \ldots \) in \( \mathbb{M}^n \) such that \( C_{i+1} \subseteq \text{Int} \ C_i \), \( i = 1, 2, \ldots \), and \( X = \bigcap_{i=1}^\infty C_i \).

For the construction of Example 6 we require the following lemma.

**Lemma 5.** If \( X \) is a cellular continuum in \( E^n \), then \( \Sigma X \) is cellular in \( E^{n+1} \).
Proof. It suffices to show that if \( U \) is an open set in \( E^{n+1} \) with 
\[ \Sigma X \subset U, \]
then there is an \((n+1)\)-cell \( C \) such that \( X \subset \operatorname{Int} C \subset C \subset U \). Since \( X \) is cellular, there exists a locally flat \( n \)-cell \( D \) in \( E^n \) such that 
\[ X \subset \operatorname{Int} D \] and \( \Sigma D \subset U \). We note that \( \Sigma X \)-(suspension points) lies in \( \operatorname{Int} (\Sigma D) \). Now, the \((n+1)\)-cell \( \Sigma D \) is locally flat since \( D \) is locally flat [9]. Staying in \( U \), we obtain \( C \) from \( \Sigma D \) by pasting a small \((n+1)\)-cell to \( \Sigma D \) at each suspension point.

We now construct some examples which give us an answer to the question raised in [12, Remark (viii)].

Example 6. There exists a noncellular \( k \)-frame in \( E^n \), \( k \geq 3 \), \( n \geq 3 \), each of whose \((k-1)\)-frames is cellular.

Construction. For \( n = 3 \) an explicit construction for such frames is given in [12, §4]. The frames are shown to be noncellular by showing that their complements fail to be simply connected.

Now suppose \( K \) is a \( k \)-frame in \( E^n \) with \( \pi_1(E^n - K) \neq 0 \) and each \((k-1)\)-frame in \( K \) cellular in \( E^n \). Let \( a \) denote the junction point of \( K \). Since \( \pi_1(E^n - K) \neq 0 \), we find that \( \pi_1(E^{n+1} - \Sigma K) \neq 0 \). Hence \( \Sigma K \) fails to be cellular. But, by Lemma 5, if \( L \) is a \((k-1)\)-frame in \( K \), then \( \Sigma L \) is cellular in \( E^{n+1} \).

We now squeeze \( \Sigma K \) to obtain the desired \( k \)-frame in \( E^{n+1} \) in two steps. First, we collapse the (tame) arc \( \Sigma a \) to a point. This collapses \( \Sigma K \) to a wedge of \( k \) disks which we denote by \( P^k \). While \( P^k \) is not cellular, its proper sub-wedges are. Now apply Corollary 4 to squeeze \( P^k \) to a \( k \)-frame. Again, the \( k \)-frame is not cellular, but its \((k-1)\)-frames are.

Remark. (iv) An easy proof of Theorem 2 (or its analog in \( E^n \), \( n \geq 5 \)) can be given in the special case when \( f \) is a suspension embedding; this would be applicable in the above construction. For this proof, we would simply note that each arc in a suspension disk can be approximated, modulo its endpoints by one whose interior is the union of a countable collection of "horizontal" and "vertical" arcs in the product structure of \( E^n = E^{n-1} \times E^1 \). Such arcs are known to be tame.

4. An example in codimension one. Seebeck shows [11] that if \( n \geq 4 \) and \( 2 \leq m \leq n - 2 \), then there is an \( m \)-cell in \( E^n \) which is wild at each of its points but each of whose arcs is tame. We show here how to use Theorem 1 to extend this result to the case \( m = n - 1 \).

Example 7. There exists an \((n-1)\)-sphere in \( E^n \), \( n \geq 3 \), which is wild at each of its points, but each of whose arcs is tame.

Construction. If \( n = 3 \), then Bing's "hooked rug" [2] provides our example. By suspending this example an appropriate number of times we obtain an \((n-1)\)-sphere \( S \) in \( E^n \) which is wild at each point and
which is the suspension of an \((n-2)\)-sphere \(S_0\) in \(E_{n-1}\). Inductively, we may suppose that each arc in \(S_0\) is tame in \(E^{n-1}\). (That \(S\) is wild at each point is shown in [10].) Let \(A\) be an arc in \(S\).

There is a dense subset \(\{t_1, t_2, t_3, \ldots\}\) of \(R\) such that \(A \cap (E^{n-1} \times \{t_i\})\) is at most 0-dimensional for each positive integer \(i\). If \(A \cap (E^{n-1} \times \{t_i\}) \neq \emptyset\), then \(A \cap (E^{n-1} \times \{t_i\})\) lies on the 2-sphere \(S \cap (E^{n-1} \times \{t_i\})\). (Unless, trivially, \(t_i = -1\) or \(1\).) But the pair \(E^{n-1} \times \{t_i\}, S \cap (E^{n-1} \times \{t_i\})\) is homeomorphic to \((E^{n-1}, S_0)\). Hence, there is an arc \(B \subset E^{n-1} \times \{t_i\}\) such that \(A \cap (E^{n-1} \times \{t_i\}) \subset B\) and \(B\) is tame in \(E^{n-1} \times \{t_i\}\). But then \(A \cap (E^{n-1} \times \{t_i\})\) is tame in \(E^{n-1} \times \{t_i\}\); hence, \(E^{n-1} \times \{t_i\} - A \cap (E^{n-1} \times \{t_i\})\) is 1-ULC. By Theorem 1, \(A\) is tame.

References

6. ———, Each disk in \(E^n\) can be squeezed to an arc (to appear).

University of Georgia, Athens, Georgia 30601