PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 30, No. 1, September 1971

SOME RADICAL PROPERTIES OF s-RINGS

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ABSTRACT. The class of s-rings includes as a proper subset the classes of associative, alternative, Lie, Jordan, and standard rings. It is shown that in any s-ring R the prime radical of R coincides with the Baer lower radical of R. Relationships between the prime radical and certain other radicals are also given.

In [6], P. Zwier defines an s-ring R to be a nonassociative ring satisfying the condition that if A is an ideal of R then A^s is also an ideal of R, where s is an integer $\geqq 2$. Also an ideal P of an s-ring R is called a prime ideal if whenever $A_1A_2 \cdots A_s \subseteq P$ for ideals A_i of R then $A_j \subseteq P$ for some j. Here $A_1A_2 \cdots A_s$ denotes the product of the ideals under all possible associations. It is announced that using these definitions one may define a prime radical and obtain results for s-rings corresponding to the results obtained by C. Tsai [5] for Jordan rings.

Specifically, if one defines the prime radical $\beta(R)$ of an s-ring R so that it coincides with the intersection of all prime ideals of the ring R and a ring R to be β-semisimple if $\beta(R) = (0)$ then one obtains the following results for any s-ring R:

1. $\beta(R) \subseteq N(R)$ where $N(R)$ denotes the nil radical of R.
2. $R/\beta(R)$ is β-semisimple.
3. R is β-semisimple if and only if R contains no nonzero nilpotent ideals.

In this note we consider the Baer lower radical of an s-ring R and show that, just as in the case of associative rings, it coincides with the prime radical. In addition we show relationships between the prime radical, a Jacobson radical studied by Brown and McCoy and a radical of Smiley.

Recall that the Baer lower radical of a ring R is defined as follows: Let N_0 be the union of all nilpotent ideals of R. Let N_1 be the union of all ideals A in R such that A/N_0 is nilpotent in R/N_0. In general if β is a limit ordinal then $N_\beta = \bigcup_{\alpha<\beta} N_\alpha$. If β has a predecessor then N_β is the union of all ideals A in R such that $A/N_{\beta-1}$ is nilpotent in $R/N_{\beta-1}$. If γ is the least ordinal such that $N_\gamma = N_{\gamma+1} = \cdots$ then N_γ

Received by the editors June 8, 1970.

AMS 1969 *subject classifications*. Primary 1710.

Key words and phrases. s-ring, prime ideal, nilpotent ideal, prime radical, Baer lower radical, Jacobson radical, semisimple.

is called the Baer lower radical of R and is denoted by $B(R)$. By the definition it is clear that $R/B(R)$ has no nonzero nilpotent ideals.

THEOREM 1. *In any s-ring R, $B(R)$ is the intersection of all ideals Q_i of R such that R/Q_i has no nonzero nilpotent ideals.*

PROOF. We may adopt the proof of this same theorem for associative rings as presented in [**2**, p. 56] since associativity is not required for that argument.

THEOREM 2. *In any s-ring R, $\beta(R)=B(R)$.*

PROOF. Let P be a prime ideal of R. Then $(\bar{0})$ is a prime ideal of $\bar{R}=R/P$ and R/P is β-semisimple. Therefore, by property 3, R/P has no nonzero nilpotent ideals. Thus $P=Q_i$ for some i and the intersection of all prime ideals in R contains $W=\bigcap Q_i$. Therefore $\beta(R) \supseteq B(R)$.

Conversely, suppose that $a \notin B(R)$. Clearly $(a)^s \not\subseteq B(R)$ for if $(a)^s \subseteq B(R)$ then $(a)/B(R)$ is a nonzero nilpotent ideal of $R/B(R)$ which is impossible. Construct the set $A=\{a_1, a_2, \cdots, a_n, \cdots\}$ as follows: $a_1=a$, $a_2 \in (a_1)^s \cap C(B(R)), \cdots, a_{n+1} \in (a_n)^s \cap C(B(R))$ where C denotes set theoretic complement. By our choice of A, $A \cap B(R)=\emptyset$. Now by Zorn's lemma the family of all ideals of R which are disjoint from A contains a maximal ideal P. We show that P is a prime ideal.

Let $P_1, P_2, \cdots, P_s$ be ideals of R such that $P_1P_2 \cdots P_s \subseteq P$. If $P_i \not\subseteq P$ for every i then $(P_i+P) \supsetneq P$ for every i. Therefore $(P_i+P)\cap A \neq \emptyset$ for every i and there exist $a_{i_1} \in P+P_1$, $a_{i_2} \in P+P_2, \cdots, a_{i_s} \in P+P_s$. Hence $(a_{i_1})(a_{i_2}) \cdots (a_{i_s}) \subseteq (P+P_1)(P+P_2) \cdots (P+P_s) \subseteq P_1P_2 \cdots P_s+P \subseteq P$. (Here (a) denotes the principal ideal generated by a.) By our choice of the a_{i_j}, $(a_{n+1}) \subseteq (a_n)$. Let

$$m = \max (i_1, i_2, \cdots, i_s).$$

Then $a_{m+1} \in (a_m)^s \subseteq (a_{i_1})(a_{i_2}) \cdots (a_{i_s}) \subseteq P$. Thus $a_{m+1} \in P$ which contradicts the fact that $A \cap P=\emptyset$. Therefore P is a prime ideal, $a \notin P$ and $\beta(R) \subseteq B(R)$.

Brown and McCoy have defined the Jacobson radical $J(R)$ of a nonassociative ring R to be the maximal quasi-regular ideal of R where $a \in R$ is said to be quasi-regular if $a \in Q(a)$ with $Q(a)$ the right ideal generated by $at-t$ with $t \in R$. They have shown (see [**1**, Theorem 8]) that $J(R)$ is v-prime for any v and that $N(R) \subseteq J(R)$. Thus $J(R)$ is a prime ideal according to the definition given here. Therefore $\beta(R) \subseteq J(R)$. Smiley [**4**] has provided a radical $S(R)$ which is the intersection of all modular maximal ideals of R. It is known [**1**, p. 254] that $J(R) \subseteq S(R)$. Therefore, for any s-ring we have the following:

THEOREM 3. *In any s-ring* R, $B(R)=\beta(R)\subseteq N(R)\subseteq J(R)\subseteq S(R)$.

Finally, it should be noted that the class of s-rings includes associative, alternative, Lie, Jordan and standard rings. The first three can very easily be shown to be 2-rings and the latter two are 3-rings as shown in [**5**, Lemma 3] and [**3**, Lemma 1].

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