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SPECTRUM OF X SATISFYING $0 \leqq X^p \leqq X$

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ABSTRACT. A sharp estimate is found for the possible spectrum of an element satisfying the polynomial inequality $0 \leqq X^p \leqq X$ in a Dedekind σ-complete real linear algebra.

1. **Statement of results.** Let $\mathfrak{A}$ be a partially ordered real linear algebra with unit, and Dedekind σ-complete (that is, a bounded increasing sequence of elements has a least upper bound).

MAIN THEOREM. *If for $X \in \mathfrak{A}$ there is an integer $p \geqq 2$ such that $0 \leqq X^p \leqq X$ then spectrum X is contained in the "spoked wheel" $\{\lambda : |\lambda| \leqq \alpha$ and $|\lambda - \lambda^p| \leqq \beta\} \cup \{\lambda : \alpha \leqq |\lambda| \leqq 1$ and λ^{p-1} positive real$\}$ where $\alpha > 0$, $\alpha = (1/p)^{(1/p-1)}$ and $\beta = \alpha - \alpha^p$. This estimate is sharp. For X a real nonnegative matrix the estimate is sharp subject to the further restriction that if $|\lambda - \lambda^p| = \beta$ for an eigenvalue λ then $(\lambda - \lambda^p)^n$ is real for some integer n.*

R. DeMarr[2] has found a sharp estimate for the real spectrum of X when $p = 2$. I. Marek has found a sharp estimate (for any p) of $R \in \lambda^{p-1}$, λ in spectrum X, for X in an ordered Banach algebra. Both results are unpublished.

2. **Order limits.**

2.1. If $Y_n \in \mathfrak{A}$ we say o-lim $Y_n = Y$ iff there exists a positive decreasing sequence X_n with greatest lower bound 0 such that $-X_n \leqq Y - Y_n \leqq X_n$ for all n. The o-limit is unique if it exists. "The o-limit of a sum is the sum of the o-limits." o-lim $\gamma Y_n = \gamma$(o-lim Y_n) for γ real. See [1] for proofs and other information on partially ordered linear algebras. If o-lim$_n \sum_{m=0}^{n} Y_m$ exists we denote it by $\sum_{m=0}^{\infty} Y_m$. For $Y_m \geqq 0$ we write $\sum_{m=0}^{\infty} Y_m < \infty$ to indicate convergence of the series.

2.2. In general, multiplication does not preserve o-limits. This explains the somewhat tricky proof of the following lemma which would be trivial for an o-continuous multiplication.

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[2] The author wishes to thank Professor DeMarr for his help and advice. The proof of 3.2 and example (b) of 3.6 are derived from his work on the case $0 \leqq X^2 \leqq X$.

LEMMA. *Suppose* $X \geqq 0$ *and* $\sum_{n=0}^{\infty} X^n < \infty$. *If* b_n *real and* $|b_n| < 1$ *then*

(a) $\sum_{n=0}^{\infty} b_n X^n$ *exists and*

(b) $X(\sum_{n=0}^{\infty} b_n X^n) = \sum_{n=0}^{\infty} b_n X^{n+1} = (\sum_{n=0}^{\infty} b_n X^n)X$.

PROOF. According to 2.1, $\sum_{n=0}^{\infty} b_n X^n = \sum_{n=0}^{\infty} b_n' X^n + \sum_{n=0}^{\infty} b_n'' X^n$ where $b_n' = \max(b_n, 0)$ and $b_n'' = \min(b_n, 0)$. Thus we need only consider the case $b_n \geqq 0$. Then (a) is trivial.

(b) $X(\sum_{n=0}^{\infty} X^n) = \sum_{n=0}^{\infty} X^{n+1}$ [1, Proposition 2, p. 640]. Define

$$Y_k = X\left(\sum_{n=0}^{\infty} b_n X^n\right) - \sum_{n=0}^{k} b_n X^{n+1} = X\left(\sum_{n=0}^{\infty} b_n X^n - \sum_{n=0}^{k} b_n X^n\right)$$

$$= X\left(\sum_{n=k+1}^{\infty} b_n X^n\right) \leqq X\left(\sum_{n=k+1}^{\infty} X^n\right) = \sum_{n=k+1}^{\infty} X^{n+1} \downarrow 0.$$

Also $Y_k \geqq -\sum_{n=k+1}^{\infty} X^{n+1}$. The right equality of (b) is proven similarly.

2.3. LEMMA. *Suppose* $X \geqq 0$, $\lambda_1 > 0$ *and* $\sum_{n=0}^{\infty} \lambda_1^{-n-1} X^n < \infty$. *If* λ *complex and* $|\lambda| = |\lambda_1|$ *then* $(\lambda - X)^{-1}$ *exists.*

Comment. The inverse is asserted to lie in the canonical complexification of $\mathfrak{A}$. That is there exist A and B in $\mathfrak{A}$ such that

$$(A + iB)(\operatorname{Re}\lambda + i\operatorname{Im}\lambda - X) = (\operatorname{Re}\lambda + i\operatorname{Im}\lambda - X)(A + iB) = 1$$

in the sense that real and imaginary parts are equal.

PROOF. Without loss of generality one may set $\lambda_1 = 1$ and $\lambda = e^{i\theta}$. Then the previous lemma shows that

$$A = \sum_{n=0}^{\infty} \cos(-n-1)\theta X^n \quad \text{and} \quad B = \sum_{n=0}^{\infty} \sin(-n-1)\theta X^n$$

converge, and further that $A + iB$ provides the desired inverse.

2.4. R. DeMarr has proven the following generalized Perron-Frobenius Theorem.

THEOREM. *Suppose the positive cone in* $\mathfrak{A}$ *is generating. Let* $X \geqq 0$ *and assume* $\Gamma = \{\lambda \text{ real}: (\lambda - X)^{-1} \geqq 0\}$ *is nonempty. Then*

(a) $0 \leqq \inf \Gamma = \mu \in$ *spectrum* X;

(b) $\lambda > \mu$ *iff* $\sum_{n=0}^{\infty} \lambda^{-n-1} X^n < \infty$;

(c) *if* λ *real,* $\lambda \in$ *spectrum* X *then* $|\lambda| \leqq \mu$.

As a corollary of 2.3 we may now add that *if* λ *complex and* $\lambda \in$ *spectrum* X *then* $|\lambda| \leqq \mu$.

3. **The Main Theorem.** Assume $0 \leqq X^p \leqq X$ for some $p \geqq 2$. We shall prove the Main Theorem by a sequence of lemmas.

3.1. LEMMA. *If $\lambda \in$ spectrum X then $|\lambda| \leqq 1$.*

PROOF. Put $Z = \sum_{k=1}^{p-1} X^k$ so that $XZ \leqq Z$. Since $X \leqq Z$ we have $X^n \leqq Z$ for $n = 1, 2, \cdots$. For $\lambda > 1$,

$$\sum_{n=0}^{\infty} \lambda^{-n-1} X^n \leqq \lambda^{-1} + \sum_{n=1}^{\infty} \lambda^{-n-1} Z < \infty.$$

Now apply 2.3.

3.2. LEMMA. *If $\eta \in$ spectrum $X - X^p$ then $|\eta| \leqq \beta$.*

PROOF. Let $Y = X - X^p$. Then $X = Y + X^p$ and

$$X^{pn} = (Y + X^p)^{pn} \geqq (n+1)\text{st term of binomial expansion}$$

$$= \binom{pn}{n} Y^{(p-1)n} X^{pn}. \tag{a}$$

Using Stirling's approximation ($n! = n^{n+1/2} e^{-n} (2\pi)^{1/2} J(n)$ where $J(n) \to 1$ as $n \to \infty$) we easily verify

$$\lim_n \binom{pn}{n}^{-1/(p-1)n} = \beta.$$

Then if $\lambda > \beta$ there exists an integer $q > 0$ and γ with $0 < \gamma < 1$ such that

$$\binom{pq}{q}^{-1} \leqq \lambda^{(p-1)q} \gamma. \tag{b}$$

Let $r = (p-1)q$. From (a) and (b) we obtain $Y^r X^{pq} \leqq \lambda^r \gamma X^{pq}$ and multiplying by $X^{(p-2)q}$,

$$Y^r X^{2r} \leqq \lambda^r \gamma X^{2r}. \tag{c}$$

Because $Y \leqq X$, $Y^{3r} \leqq \lambda^r \gamma X^{2r}$, and continuing by induction we see that $Y^{nr} \leqq \lambda^{(n-2)r} \gamma^{n-2} X^{2r}$ for $n \geqq 3$. Then

$$\sum_{n=0}^{\infty} \lambda^{-nr-1} Y^{nr} \leqq \sum_{n=0}^{2} \lambda^{-nr-1} Y^{nr} + \sum_{n=3}^{\infty} \lambda^{-2r-1} \gamma^{n-2} X^{2r} < \infty. \tag{d}$$

Multiplying (d) successively by the first $r-1$ powers of $\lambda^{-1} Y$ and adding we obtain $\sum_{n=0}^{\infty} \lambda^{-n-1} Y^n < \infty$. Now apply 2.3.

3.3. LEMMA. *If* $\lambda \in$ *spectrum* X *then* $|\lambda - \lambda^p| \leqq \beta$.

PROOF. By the spectral mapping theorem, $\lambda - \lambda^p \in$ spectrum $X - X^p$.

3.4. LEMMA. *If* $\lambda \in$ *spectrum* X *and* $|\lambda| = \alpha$ *then* $\lambda^{p-1} = \alpha^{p-1}$.

PROOF. $\alpha(1-\alpha^{p-1}) = \beta \geqq |\lambda - \lambda^p| = |\lambda| \, |1-\lambda^{p-1}| = \alpha |1 - \lambda^{p-1}|$. Thus $1-\alpha^{p-1} \geqq |1-\lambda^{p-1}|$. Since $|\lambda^{p-1}| = \alpha^{p-1}$ the conclusion follows.

3.5. LEMMA. *If* $\lambda \in$ *spectrum* X *and* $|\lambda| \geqq \alpha$ *then* λ^{p-1} *is positive real.*

PROOF. $0 \leqq (\alpha|\lambda|^{-1}X)^p \leqq \alpha|\lambda|^{-1}X$. $\alpha|\lambda|^{-1}\lambda \in$ spectrum $\alpha|\lambda|^{-1}X$. $|\alpha|\lambda|^{-1}\lambda| = \alpha$, so, by 3.4, $\alpha^{p-1}|\lambda|^{-(p-1)}\lambda^{p-1} = \alpha^{p-1}$.

3.6. Combining 3.1, 3.3 and 3.5 we have proven our Main Theorem. Sharpness: (a) Let Y be the $(p-1)\times(p-1)$ permutation matrix with $(Y)_{ij} = 1$ if $i = j+1$ or if $i = 1$ and $j = p-1$, $(Y)_{ij} = 0$ otherwise. Let $X = kY$ for $0 \leqq k \leqq 1$. $X^{p-1} = k^{p-1}I$. $0 \leqq X^p \leqq X$. $\lambda \in$ spectrum X iff $\lambda^{p-1} = k^{p-1}$.

(b) Let $E = \{\lambda : |\lambda| \leqq \alpha,\ |\lambda - \lambda^p| \leqq \beta\}$. Let $\mathcal{A}$ be the set of pairs (k, f) where k is real and f is a bounded complex function on E. Define pointwise addition and multiplication. Define $(k, f) \leqq (j, g)$ iff $|g(\lambda) - f(\lambda)| \leqq j - k$ for all $\lambda \in E$. Let $X = (\alpha, f)$ where $f(\lambda) = \lambda$. $0 \leqq X^p \leqq X$ and spectrum $X = E$.

(c) The mapping $z = f(w) = w - w^p$ maps E one-to-one onto the disc $|z| \leqq \beta$. Using implicit differentiation one can see that the inverse function $w = g(z)$ can be expressed as a series $\sum_{n=1}^{\infty} b_n z^n$ with positive coefficients and convergent in all of $|z| \leqq \beta$.

If Y is a nonnegative matrix with maximum eigenvalue $\leqq \beta$, then $X = \sum_{n=1}^{\infty} b_n Y^n$ converges and satisfies $X - X^p = Y$. Conversely, by 3.2, if $0 \leqq X^p \leqq X$, then the maximum eigenvalue of $X - X^p$ is $\leqq \beta$.

We conclude that the part of spectrum X in E may contain exactly those λ such that $\lambda - \lambda^p$ is an eigenvalue of some positive matrix with maximum eigenvalue $\leqq \beta$. The only restrictions on such an eigenvalue is the one described in our Main Theorem (see [**3**, pp. 280, 286]).

3.7. By small modifications in the proof of the Main Theorem we can obtain this sharp estimate:

If $0 \leqq X$ and $0 \leqq X^p \leqq X^q$ for $0 < q < p$ then spectrum $X \subseteq \{\lambda : |\lambda| \leqq \alpha,\ |\lambda^q - \lambda^p| \leqq \beta\} \cup \{\lambda : \alpha \leqq |\lambda| \leqq 1,\ \lambda^{(p-q)} \text{ positive real}\}$ where $\alpha > 0$, $\alpha = (q/p)^{1/(p-q)}$ and $\beta = \alpha^q - \alpha^p$. (For the modification of the proof of 3.2, let $X^q = Y + X^p$ and look at the $(qn+1)$st term of the binomial expansion of $(X^q)^{pn}$.)

References

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