NOTE ON THE KLOOSTERMAN SUM

KENNETH S. WILLIAMS

Abstract. The Kloosterman sum

\[ \sum_{x=0: (x, p) = 1}^{p-1} \exp(2\pi i x(x + \bar{x})/p^a), \]

where \( p \) is an odd prime, \( \alpha \geq 2 \) and \( (n, p) = 1 \), is evaluated in a very short direct way.

Let \( p \) denote an odd prime, \( \alpha \) a positive integer, and \( n \) an integer such that \( (n, p) = 1 \). The Kloosterman sum \( A_{p^a}(n) \) is given by

\[ A_{p^a}(n) = \sum_{x=0}^{p^{a-1}} \exp(2\pi i x(x + \bar{x})/p^a), \]

where the dash (') indicates that \( x \) only takes values from 0, 1, \( \ldots, p^\alpha - 1 \) which are coprime with \( p \), and \( \bar{x} \) is the unique solution of the congruence \( x \bar{x} \equiv 1 \) (mod \( p^\alpha \)) satisfying \( 0 < \bar{x} < p^\alpha \). Salie [3] has evaluated \( A_{p^a}(n) \) explicitly when \( \alpha \geq 2 \). His evaluation is based upon induction. A direct (but fairly long) proof has been given by Whiteman [4] which requires results concerning Ramanujan sums. In this note we give a modification of Salie’s original argument which gives a very short direct evaluation of \( A_{p^a}(n) \). (The referee has kindly pointed out that essentially the same technique has been used by Estermann [2], Carlitz [1].)

We let \( \gamma = \alpha - \lfloor \alpha/2 \rfloor \) and \( \delta = \lceil \alpha/2 \rceil \), so that \( \alpha = \gamma + \delta \), \( 2 \gamma \geq \alpha \) and \( \gamma \geq \delta \geq 1 \). Setting \( x = u + v p^\gamma \) \((u = 0, 1, \ldots, p^\gamma - 1; v = 0, 1, \ldots, p^\delta - 1)\) in (1), so that \( \bar{x} = \bar{u} - \bar{u} v p^\gamma \) (mod \( p^a \)), we obtain

\[ A_{p^a}(n) = \sum_{u=0}^{p^{\gamma-1}} \sum_{v=0}^{p^{\delta-1}} \exp(2\pi i ((u + v p^\gamma) + (\bar{u} - \bar{u} v p^\gamma))/p^a) \]

\[ = \sum_{u=0}^{p^{\gamma-1}} \exp(2\pi i (u + \bar{u})/p^a) \sum_{v=0}^{p^{\delta-1}} \exp(2\pi i v (1 - \bar{u}^2)/p^\delta) \]

\[ = p^\delta \sum_{u=0: u^2 \equiv 1 \pmod{p^\delta}}^{p^{\gamma-1}} \exp(2\pi i (u + \bar{u})/p^a). \]

If \( \alpha \) is even, say \( \alpha = 2\beta \), then \( \gamma = \delta = \beta \), and as the solutions \( u \) of \( u^2 \equiv 1 \) (mod \( p^\delta \)) in the range \( 0 \leq u \leq p^\delta - 1 \) are \( u = 1, p^\delta - 1 \) (so that...
Let \( \alpha = 1, p^{2\beta} - p^\beta - 1 \) respectively, we have

\[
A_{p^{2\beta}}(n) = p^\beta \left\{ \exp\left(\frac{4\pi in}{p^{2\beta}}\right) + \exp\left(-\frac{4\pi in}{p^{2\beta}}\right) \right\} = 2p^\beta \cos\left(\frac{4\pi n}{p^{2\beta}}\right).
\]

If \( \alpha \) is odd, say \( \alpha = 2\beta + 1 \), then \( \gamma = \beta + 1 \), \( \delta = \beta \), and as the solutions \( u \) of \( u^2 \equiv 1 \pmod{p^\beta} \) in the range \( 0 \leq u \leq p^{\beta+1} - 1 \) are \( u = 1 + wp^\beta \) \( (w = 0, 1, \ldots, p-1) \), \( -1 + wp^\beta \) \( (w = 1, 2, \ldots, p) \) (so that \( u = 1 - wp^\beta + w^2p^{2\beta} \), \( -1 - wp^\beta - w^2p^{2\beta} \) (mod \( p^{2\beta+1} \)) respectively) we have

\[
A_{p^{2\beta+1}}(n) = p^\beta \left\{ \exp\left(\frac{4\pi in}{p^{2\beta+1}}\right) \sum_{w=0}^{p-1} \exp\left(2\pi inw^2/p\right) \right\}
\]

\[
\quad + \exp\left(-\frac{4\pi in}{p^{2\beta+1}}\right) \sum_{w=1}^{p} \exp\left(-2\pi inw^2/p\right)
\]

\[
= 2(n/p)p^{\beta+1/2}\cos\left(\frac{4\pi n}{p^{2\beta+1}}\right), \quad \text{if} \quad p \equiv 1 \pmod{4},
\]

\[
= -2(n/p)p^{\beta+1/2}\sin\left(\frac{4\pi n}{p^{2\beta+1}}\right), \quad \text{if} \quad p \equiv 3 \pmod{4},
\]

using the well-known result [4]

\[
\sum_{w=0}^{p-1} \exp\left(2\pi inw^2/p\right) = \frac{n}{p}i^{(p-1)/4}p^{1/2}.
\]

REFERENCES


Carleton University, Ottawa, Ontario, Canada