PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 30, No. 1, September 1971

EMBEDDING THE DUAL OF Π_m IN THE LATTICE OF EQUATIONAL CLASSES OF COMMUTATIVE SEMIGROUPS

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ABSTRACT. The lattice of equational classes of commutative semigroups does not satisfy any special lattice laws.

In [4] R. Schwabauer proved that the lattice $\mathcal{L}$ of equational classes of commutative semigroups is nonmodular. In this paper we will prove a maximal extension of this result, namely, *$\mathcal{L}$ does not satisfy any special lattice laws.*

The free commutative semigroup on countably many generators, $\mathfrak{F}(\omega)$, is the set of sequences $(u_n)_{n\in N}$ of nonnegative integers, such that $u_n=0$ for all but finitely many $n\in N$ and $\sum u_n\geqq 1$, with component-wise addition. [For convenience, we write (u_n) for $(u_n)_{n\in N}$.]

A commutative semigroup equation is a pair $((u_n), (v_n))$ of elements of $\mathfrak{F}(\omega)$ (see [1]). A commutative semigroup $(\mathcal{S}, \cdot)$ satisfies the equation $((u_n), (v_n))$ iff, for every family (a_n) of elements of $\mathcal{S}$, $\Pi\{a_n^{u_n}\mid u_n\neq 0\} = \Pi\{a_n^{v_n}\mid v_n\neq 0\}$.

A set Σ of equations is closed [1, p. 170, Definition 2] iff it contains all trivial equations, is symmetric and transitive, and is closed under multiplication and substitution of terms for variables. Thus Σ is closed iff it satisfies the following conditions:

(P1): $((u_n), (u_n))\in\Sigma$ for all $(u_n)\in\mathfrak{F}(\omega)$.

(P2): If $((u_n), (v_n))\in\Sigma$ then $((v_n), (u_n))\in\Sigma$.

(P3): If $((u_n), (v_n))\in\Sigma$ and $((v_n), (w_n))\in\Sigma$ then $((u_n), (w_n))\in\Sigma$.

(P4): If $((u_n), (v_n))\in\Sigma$ and $((u_n'), (v_n'))\in\Sigma$ then $((u_n+u_n'), (v_n+v_n'))\in\Sigma$.

(P5): If $((u_n), (v_n))\in\Sigma$, $(k_n)\in\mathfrak{F}(\omega)$ and $p\in N$, then the result of "substituting (k_n) for the pth variable" in $((u_n), (v_n))$ is in Σ, i.e., $((w_n), (x_n))\in\Sigma$ where $w_n=u_n+k_nu_p$ for $n\neq p$, $w_p=k_pu_p$, and $x_n=v_n+k_nv_p$ for $n\neq p$ and $x_p=k_pv_p$.

(Note that these conditions (P1) to (P5) are, essentially, a restatement of conditions (i) to (v) in Grätzer [1, p. 170, Definition 2]; in condition (iv) we need only consider the one binary operation, hence

Received by the editors May 14, 1970.

AMS 1969 *subject classifications*. Primary 2090, 0630; Secondary 0830.

Key words and phrases. Equation, equational class, commutative semigroup, lattice, partition lattice, lattice law.

[1] Research supported by NRC Grant A7256.

[2] Research supported by NRC Grant A2985.

the different form of (P4).) $\Gamma(\Sigma)$ will denote the deductive closure of Σ.

Let $\mathcal{L}'$ be the lattice of closed sets of commutative semigroup equations; then $\mathcal{L}'$ is dually isomorphic to the lattice $\mathcal{L}$ of equational classes of commutative semigroups.

For each $m\in N$, let Π_m be the partition lattice on $\{1, 2, \cdots, m\}$.

THEOREM 1. *For each* $m\in N$, Π_m *is (isomorphic to) a sublattice of* $\mathcal{L}'$.

PROOF. Let m be a fixed natural number. If $\pi\in\Pi_m$, we write $\equiv_\pi$ for the equivalence relation on $\{1, 2, \cdots, m\}$ induced by π. Let $\Sigma=\{((u_n), (v_n)) \mid \sum u_n, \sum v_n\geqq 2m+2\}\cup\{((u_n), (u_n)) \mid (u_n)\in\mathfrak{F}(\omega)\}$. It is clear from (P1) to (P5) that Σ is a closed set of equations. For each $\pi\in\Pi_m$, define a set $\Sigma(\pi)$ of equations as follows: $((u_n),(v_n))\in\Sigma(\pi)$ iff there exist j, k with $u_n=v_n=0$ for all $n\neq j$, k, $u_j+u_k=2m+1=v_j+v_k$, and either $u_j\equiv_\pi v_j$ or $u_k\equiv_\pi v_k$. Note that if $u_j+u_k=2m+1$ and $u_j\equiv_\pi v_j$ then $u_k>m$, thus u_k is not equivalent to anything modulo π. Then, since π is a partition, it follows that if $((u_n), (v_n))\in\Sigma(\pi)$ and if $((v_n), (w_n))\in\Sigma(\pi)$, then $((u_n), (w_n))\in\Sigma(\pi)$. $\Sigma(\pi)$ is obviously symmetric. Applying (P4) to two equations in $\Sigma(\pi)$ yields an equation in Σ. Applying (P5) with $\sum k_n=1$ to an equation in $\Sigma(\pi)$ yields either a trivial equation or an equation in $\Sigma(\pi)$; applying (P5) with $\sum k_n\geqq 2$ and with $u_p\geqq 1$ yields an equation in Σ; and applying (P5) with $u_p=0$ does not change the equation. Thus $\Sigma\cup\Sigma(\pi)$ is a closed set of equations.

For two partitions π_1, π_2, if $\pi_1\wedge\pi_2$ and $\pi_1\vee\pi_2$ are the meet and join of π_1 and π_2 in Π_m then

$$(\Sigma\cup\Sigma(\pi_1))\cap(\Sigma\cup\Sigma(\pi_2)) = \Sigma\cup(\Sigma(\pi_1)\cap\Sigma(\pi_2)) = \Sigma\cup(\Sigma(\pi_1\wedge\pi_2)).$$

Also it is clear that

$$\begin{aligned}(\Sigma\cup\Sigma(\pi_1))\vee_{\mathcal{L}'}(\Sigma\cup\Sigma(\pi_2)) &= \Gamma(\Sigma\cup\Sigma(\pi_1)\cup\Sigma(\pi_2))\\ &\subseteq \Sigma\cup\Sigma(\pi_1\vee\pi_2).\end{aligned}$$

Conversely, if $((u_n), (v_n))\in\Sigma(\pi_1\vee\pi_2)$, then there exist j, k with $u_n=v_n=0$ for all $n\neq j$, k, $u_k+u_j=v_k+v_j=2m+1$ and, w.l.o.g. $u_j\equiv_{\pi_1\vee\pi_2} v_j$. But then there exist $w_1, \cdots, w_p$ in $\{1, 2, \cdots, m\}$ such that $w_1=u_j$, $w_p=v_j$, and $w_i\equiv_{\pi_1} w_{i+1}$ for i odd, $w_i\equiv_{\pi_2} w_{i+1}$ for i even. Let $\alpha_i\in\mathfrak{F}(\omega)$ have jth entry w_i, kth entry $2m+1-w_i$, and all other entries zero. Then $\alpha_1=(u_n)$, $\alpha_p=(v_n)$ and $(\alpha_i, \alpha_{i+1})\in\Sigma(\pi_1)$ for i odd and $(\alpha_i, \alpha_{i+1})\in\Sigma(\pi_2)$ for i even. It follows that

$$((u_n),(v_n)) = (\alpha_1, \alpha_p) \in \Gamma(\Sigma(\pi_1)\cup\Sigma(\pi_2)).$$

Hence

$$\Sigma \cup \Sigma(\pi_1 \vee \pi_2) \subseteq \Gamma(\Sigma(\pi_1) \cup \Sigma(\pi_2) \cup \Sigma).$$

Thus we have

$$(\Sigma \cup \Sigma(\pi_1)) \wedge_{\mathfrak{L}'} (\Sigma \cup \Sigma(\pi_2)) = \Sigma \cup \Sigma(\pi_1 \wedge \pi_2)$$

and

$$(\Sigma \cup \Sigma(\pi_1)) \vee_{\mathfrak{L}'} (\Sigma \cup \Sigma(\pi_2)) = \Sigma \cup \Sigma(\pi_1 \vee \pi_2).$$

If follows that the mapping $\pi \rightarrow \Sigma \cup \Sigma(\pi)$ is a homomorphism of Π_m into $\mathfrak{L}'$.

It is clear that if $\pi_1 \neq \pi_2$ then $\Sigma \cup \Sigma(\pi_1) \neq \Sigma \cup \Sigma(\pi_2)$; thus this homomorphism is one-to-one, and this yields the desired result.

THEOREM 2. *$\mathfrak{L}$ does not satisfy any special lattice laws.*

PROOF. From D. Sachs [3] it is known that the family of partition lattices Π_m, $m = 1, 2, \cdots$, does not satisfy any special lattice laws.

CONCLUDING REMARK. One might consider the possibility of embedding the dual of Π_∞ into $\mathfrak{L}$, but a recent paper of P. Perkins [2] shows that this is impossible because $\mathfrak{L}$ is countable, whereas Π_∞ is uncountable.

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