ON THE IMMERSION OF AN n-DIMENSIONAL MANIFOLD IN n+1-DIMENSIONAL EUCLIDEAN SPACE

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Abstract. Consider the subset of n+1-dimensional Euclidean space swept out by the tangent hyperplanes drawn through the points of an immersed compact closed connected n-dimensional smooth manifold. If this is not all of the Euclidean space, then the manifold is diffeomorphic to a sphere, the immersion is an embedding, the image of the immersion is the boundary of a unique open starshaped set, and the set of points not on any tangent hyperplane is the interior of the kernel of the open starshaped set. A converse statement also holds.

Let \( M \) be a compact closed connected \( n \)-dimensional (\( n \geq 2 \)) smooth (infinitely differentiable) manifold and \( \varphi: M \rightarrow \mathbb{R}^{n+1} \) a smooth immersion of \( M \) into \( n+1 \)-dimensional Euclidean space. For each \( p \in M \) consider the hyperplane \( T_p \) in \( \mathbb{R}^{n+1} \) drawn through \( \varphi(p) \) and tangent to \( \varphi(M) \). We prove that if \( \cup_{p \in M} T_p \neq \mathbb{R}^{n+1} \), then \( M \) is diffeomorphic to the \( n \)-sphere, \( \varphi \) is actually an embedding, there exists a unique open starshaped set \( V \subset \mathbb{R}^{n+1} \) such that \( \partial V = \varphi(M) \), and \( \mathbb{R}^{n+1} - \cup_{p \in M} T_p = \text{int}(\text{kernel } V) \), where kernel \( V = \{ p \in V \mid t \varphi + (1-t)q \in V \text{ for all } q \in V \text{ and } 0 \leq t \leq 1 \} \). Conversely, if \( \varphi(M) = \partial V \) for some open starshaped set \( V \subset \mathbb{R}^{n+1} \) with \( \text{int}(\text{kernel } V) \neq \emptyset \), then \( \cup_{p \in M} T_p \neq \mathbb{R}^{n+1} \).

Notation. We denote the tangent space to \( M \) at \( p \) by \( T_M \) and the induced tangent space map of \( \varphi \) by \( d\varphi |_p \). For \( x = (x_1, \ldots, x_n) \in \mathbb{R}^n \) let \( ||x|| = (x_1^2 + \cdots + x_n^2)^{1/2} \). For \( p, q \in \mathbb{R}^n \) define \( [p, q] = \{(1-x)p + xq \mid 0 \leq x < 1 \} \) and define \( [p, q]_1, (p, q], \) and \( (p, q) \) similarly. If \( A \subset \mathbb{R}^n \) then \( \partial A, \text{int } A, \) and \( \text{cl } A \) will denote the topological boundary, interior, and closure of \( A \).

Proof. We may suppose that \( 0 \in \cup_{p \in M} T_p \). Note that \( \varphi(p) \in T_p \) for all \( p \in M \) and so \( 0 \in \varphi(M) \). Consider the differentiable map \( \varphi: \mathbb{R}^{n+1} - \{0\} \rightarrow S^n \) defined by \( \varphi(x) = x/||x|| \). It is intuitively obvious and easy to verify analytically that \( d\varphi |_x v = 0 \) iff \( v = \lambda x \) for some \( \lambda \in \mathbb{R} \). From \( 0 \in \cup_{p \in M} T_p \) it follows that each \( v \in d\varphi |_p T_M = T_p - \varphi(p), v \neq 0 \), is not of the form \( v = \lambda \varphi(p), \lambda \in \mathbb{R} \). For otherwise \( \lambda \neq 0 \) and \( -\lambda^{-1}v = -\varphi(p) \)

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\[ d(\varphi \circ \sigma) \big|_p = d\varphi \big|_{\sigma(p)} \circ d\sigma \big|_p \]

is 1-1 and thus an isomorphism onto \( TS^n(\varphi(\sigma(p))) \) for each \( p \in M \). It follows from the implicit function theorem that for each \( p \in M \), \( \varphi \circ \sigma \) maps some open neighborhood of \( p \) diffeomorphically onto an open neighborhood of \( (\varphi \circ \sigma)(p) \in S^n \).

Hence for each \( q \in S^n \), \( (\varphi \circ \sigma)^{-1} \) is a discrete space. But \( (\varphi \circ \sigma)^{-1}(q) \) is a closed subspace of the compact space \( M \) and thus \( (\varphi \circ \sigma)^{-1}(q) \) is compact; it follows that \( (\varphi \circ \sigma)^{-1}(q) \) must be a finite set. Let \( (\varphi \circ \sigma)^{-1}(q) = \{ p_1, \ldots, p_N \} \) with \( p_i \neq p_j \) for \( i \neq j \). From above, we know that for each \( i \), \( 1 \leq i \leq N \), there is an open neighborhood \( U_i \) of \( p_i \) which is mapped diffeomorphically onto an open neighborhood \( V_i \) of \( q \). Since \( M \) is Hausdorff the \( U_i \) can be chosen disjoint. Then

\[
V = \bigcap_{i=1}^{N} V_i \cap \left( S^n - (\varphi \circ \sigma) \left( M - \bigcup_{i=1}^{N} U_i \right) \right)
\]

is an open neighborhood of \( q \) such that \( (\varphi \circ \sigma)^{-1}(V) \) is a disjoint union of open sets each one of which is mapped diffeomorphically (and thus homeomorphically) onto \( V \). It follows that \( (\varphi \circ \sigma)(M) \) is an open subset of \( S^n \). But since \( M \) is compact, \( (\varphi \circ \sigma)(M) \) is also compact and thus closed and because \( S^n \) is connected we must have \( (\varphi \circ \sigma)(M) = S^n \). This shows that \( \varphi \circ \sigma : M \to S^n \) is a covering map and because \( M \) is connected and \( S^n \) (\( n \geq 2 \)) is simply connected, \( \varphi \circ \sigma \) must be a homeomorphism; hence 1-1 and hence a diffeomorphism. This establishes assertion (1).

Since \( \varphi \circ \sigma \) is 1-1, \( \varphi \) must be 1-1. Hence \( \sigma \) is an embedding (\( M \) is compact) and assertion (2) is established.

Consider the set \( V = \bigcup_{p \in M} \{ 0, \sigma(p) \} \). Clearly \( V \) is starshaped with 0 as a star-center. Set \( W = \bigcup_{p \in M} \{ t\sigma(p) \mid t > 1 \} \). Since \( \varphi \circ \sigma \) is a homeomorphism it follows that \( W \) is connected and \( V \cup W = \mathbb{R}^{n+1} - \varphi(M) \).

By the Jordan-Brouwer separation theorem \( \mathbb{R}^{n+1} - \varphi(M) \) consists of two open components, one bounded, \( X \), and one unbounded, \( Y \), such that \( \text{bdry } X \cup \varphi(M) = \text{bdry } Y \). Since \( V \) and \( W \) are connected, disjoint, \( V \cup W = \mathbb{R}^{n+1} - \varphi(M) \), and \( W \) is unbounded, it follows that \( V = X \) and \( W = Y \). Hence \( V \) is open and \( \text{bdry } V = \varphi(M) \).

We will now show that \( V \) is the only open starshaped set whose point-set boundary equals \( \varphi(M) \). Suppose \( V' \) is an open starshaped set such that \( \text{bdry } V' = \varphi(M) \). Since \( V' \) is starshaped it is connected and hence either \( V' \subset V \) or \( V' \subset W \). Now using the easily established facts that \( V' \) and \( \mathbb{R}^{n+1} - \text{cl } V' \) are open, \( V' \cup (\mathbb{R}^{n+1} - \text{cl } V') = \mathbb{R}^{n+1} \).
-\mathcal{g}(M), \mathbb{R}^{n+1} - \text{cl } V' \text{ is unbounded}, V \text{ and } W \text{ are the connected components of } \mathbb{R}^{n+1} - \mathcal{g}(M), \text{ it follows that } V' = V.

We have seen that an arbitrary point of \( Z = \mathbb{R}^{n+1} - \bigcup_{p \in M} T_p \) (which we took to be 0 by shifting the origin if necessary) is a star-center for \( V \), i.e. is in kernel \( V \). Hence \( Z \subseteq \text{kernel } V \). We will establish \( Z \subseteq \text{int}(\text{kernel } V) \) by showing that \( Z \) is open. To show this we merely have to show that an arbitrary point of \( Z \) (which we again take to be 0) is in an open set \( N \) contained in \( Z \), \( 0 \in N \subseteq Z \). Consider the map \( k: TM \to \mathbb{R}^{n+1} \) given by

\[
k(v) = \mathcal{g}(\pi(v)) + \left. d\mathcal{g} \right|_{\pi(v)}(v)
\]

for all \( v \in TM \) where \( TM \equiv \text{the tangent bundle of } M \) and \( \pi: TM \to M \) is the canonical projection. Then \( Z = \mathbb{R}^{n+1} - k(TM) \). Set \( l = \sup_{p \in M} \left\| \mathcal{g}(p) \right\| \) which is finite since \( M \) is compact. Because \( k \) is continuous \( k^{-1}(B_1) \) is closed where \( B_1 = \{ x \in \mathbb{R}^{n+1} | \left\| x \right\| \leq 1 \} \). We may pull back a Riemannian metric via \( \mathcal{g} \), i.e. we can define an inner product for all pairs of vectors \( v, w \) such that \( \pi(v) = \pi(w) \) by setting \( (v, w) = \left( d\mathcal{g} \right|_{\pi(v)}(v), d\mathcal{g} \right|_{\pi(v)}(w) \) and this inner product is clearly continuous. Set \( \left\| v \right\| = \sqrt{(v, v)} \) for each \( v \in TM \). Note that if \( \left\| v \right\| > 2l \) then

\[
\left\| k(v) \right\| = \left\| \mathcal{g}(\pi(v)) + \left. d\mathcal{g} \right|_{\pi(v)}(v) \right\|
\geq \left\| \left. d\mathcal{g} \right|_{\pi(v)}(v) \right\| - \left\| \mathcal{g}(\pi(v)) \right\|
\geq \left\| v \right\| - l > 2l - l = l.
\]

This shows that \( k^{-1}(B_1) \subseteq \{ v \in TM | \left\| v \right\| \leq 2l \} = Q \). But \( Q \) is compact since \( M \) is compact and the Tychonoff theorem is easily seen to carry over to this bundle situation. Hence, because \( k^{-1}(B_1) \) is closed it is also compact. Thus \( k(k^{-1}(B_1)) = B_1 \cap k(TM) \) is compact and hence closed. Finally

\[
0 \in N = \text{int } B_1 - k(TM) = \text{int } B_1 - B_1 \cap k(TM) \subset \mathbb{R}^{n+1} - k(TM) = Z
\]

and \( N = \text{int } B_1 - B_1 \cap k(TM) \) is clearly open. Hence \( Z \) is open and \( Z \subseteq \text{int} \text{kernel } V \) as we wished to show.

Next we will establish the converse inclusion \( Z \supseteq \text{int} \text{kernel } V \). Consider an arbitrary point of \( \text{int} \text{kernel } V \) which we may take to be 0. Next take an arbitrary point \( p \) of \( M \).

It follows from \( V \) being starshaped that \( \text{cl } V \) is also star-shaped. It is also easy to show that \( \text{kernel} \text{cl } V \supseteq \text{kernel } V \). Since \( 0 \in \text{int} \text{kernel } V \) there is an \( \epsilon > 0 \) such \( D_\epsilon = \{ x \in \mathbb{R}^{n+1} | \left\| x \right\| < \epsilon \} \subset \text{kernel } V \). Then \( C = \bigcup_{\|x\| < \epsilon} \{ x, \mathcal{g}(p) \} \subseteq \text{cl } V \). Actually \( C \subseteq V \). To see
this, note first that \( C = \bigcup_{0 < \alpha < 1} \alpha(D_\omega + (1 - \alpha)\partial(p)) \) and is thus open. Secondly, it is easily seen that \( \text{int}(\text{cl } V) = V \). So we have \( C \subseteq \text{int}(\text{cl } V) = V \) as claimed.

Now suppose \( 0 \in Z \). Then \( -\partial(p) \in \partial(V) \) and so there is a curve \( \gamma \) in \( M \) such that \( \gamma(0) = p \) and the tangent to \( \gamma \circ \gamma \) at \( 0 \) is \( -\partial(p) \). It then follows that \( (\gamma \circ \gamma)(t) \in C \) for sufficiently small positive \( t \). But this contradicts \( C \subset V \) and \( \partial(M) = \partial \bigcup_{r \in \mathbb{R}^{n+1}} V \). Hence we must have \( 0 \in Z \) as we desired. Therefore \( \text{int}(\text{kernel } V) \subset Z \) and so \( \text{int}(\text{kernel } V) = Z \) is established.

Finally we will establish the converse assertion. Suppose \( \partial(M) = \partial \bigcup_{r \in \mathbb{R}^{n+1}} V \) for some open starshaped set \( V \subset \mathbb{R}^{n+1} \) such that \( \text{int}(\text{kernel } V) \neq \emptyset \). We wish to show that \( Z \neq \emptyset \).

Without loss of generality we may assume \( 0 \in \text{int}(\text{kernel } V) \). It is now sufficient to show that \( 0 \in \partial(V) \) for each \( p \in M \). Take a \( p \in M \). Since \( 0 \in \text{int}(\text{kernel } V) \) there is an \( \epsilon > 0 \) such that \( \{ x \in \mathbb{R}^{n+1} \mid \|x\| < \epsilon \} \subset \text{kernel } V \). Consider the set \( C = \bigcup_{0 < \alpha < 1} \{ x, \partial(p) \} \). We will show that \( C \subset V \).

Let \( y = (1 - \alpha)x + \alpha \partial(p), \ 0 \leq \alpha < 1, \ \|x\| < \epsilon \), be an arbitrary point of \( C \). Since \( \partial(p) \in \partial \bigcup_{r \in \mathbb{R}^{n+1}} V \) there is a sequence \( x_m \) in \( V \) such that \( x_m \to \partial(p) \). The sequence \( y_m = (1 - \alpha)^{-1}(y - \alpha x_m) = x + (1 - \alpha)^{-1}(\partial(p) - x_m) \to x \) and hence \( \|y_m\| < \epsilon \) for some \( m \). Then \( y_m \in \text{kernel } V \) and so \( y = (1 - \alpha)y_m + \alpha x_m \in V \). Hence \( C \subset V \) as we wished to show.

Now the “curve argument” in the last paragraph of the proof of \( \text{int}(\text{kernel } V) = Z \) can be used here and it gives the desired conclusion, \( 0 \in \partial(V) \). Hence \( 0 \in Z \neq \emptyset \). This completes the proof of the theorem.

**Bibliography**


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