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ON THE IMMERSION OF AN n-DIMENSIONAL MANIFOLD IN $n+1$-DIMENSIONAL EUCLIDEAN SPACE

BENJAMIN HALPERN

ABSTRACT. Consider the subset of $n+1$-dimensional Euclidean space swept out by the tangent hyperplanes drawn through the points of an immersed compact closed connected n-dimensional smooth manifold. If this is not all of the Euclidean space, then the manifold is diffeomorphic to a sphere, the immersion is an embedding, the image of the immersion is the boundary of a unique open starshaped set, and the set of points not on any tangent hyperplane is the interior of the kernel of the open starshaped set. A converse statement also holds.

Let M be a compact closed connected n-dimensional ($n \geqq 2$) smooth (infinitely differentiable) manifold and $\mathcal{I}: M \to \boldsymbol{R}^{n+1}$ a smooth immersion of M into $n+1$-dimensional Euclidean space. For each $p \in M$ consider the hyperplane T_p in $\boldsymbol{R}^{n+1}$ drawn through $\mathcal{I}(p)$ and tangent to $\mathcal{I}(M)$. We prove that if $\bigcup_{p\in M} T_p \neq R^{n+1}$, then M is diffeomorphic to the n-sphere, $\mathcal{I}$ is actually an embedding, there exists a unique open starshaped set $V \subset R^{n+1}$ such that bdry $V = \mathcal{I}(M)$, and $R^{n+1} - \bigcup_{p\in M} T_p = \operatorname{int}(\text{kernel } V)$, where kernel $V = \{p \in V \mid tp + (1-t)q \in V \text{ for all } q \in V \text{ and } 0 \leqq t \leqq 1\}$. Conversely, if $\mathcal{I}(M) = \text{bdry } V$ for some open starshaped set $V \subset R^{n+1}$ with $\operatorname{int}(\text{kernel } V) \neq \emptyset$, then $\bigcup_{p\in M} T_p \neq R^{n+1}$.

NOTATION. We denote the tangent space to M at p by TM_p and the induced tangent space map of $\mathcal{I}$ by $d\mathcal{I}|_p$. For $x = (x_1, \cdots, x_n) \in \boldsymbol{R}^n$ let $\|x\| = (x_1^2 + \cdots + x_n^2)^{1/2}$. For $p, q \in \boldsymbol{R}^n$ define $[p, q) = \{(1-x)p + xq \mid 0 \leqq x < 1\}$ and define $[p, q]$, $(p, q]$, and (p, q) similarly. If $A \subset \boldsymbol{R}^n$ then bdry A, int A, and cl A will denote the topological boundary, interior, and closure of A.

PROOF. We may suppose that $0 \notin \bigcup_{p\in M} T_p$. Note that $\mathcal{I}(p) \in T_p$ for all $p \in M$ and so $0 \notin \mathcal{I}(M)$. Consider the differentiable map $\mathcal{P}: \boldsymbol{R}^{n+1} - \{0\} \to S^n$ defined by $\mathcal{P}(x) = x/\|x\|$. It is intuitively obvious and easy to verify analytically that $d\mathcal{P}|_x v = 0$ iff $v = \lambda x$ for some $\lambda \in \boldsymbol{R}$. From $0 \notin \bigcup_{p\in M} T_p$ it follows that each $v \in d\mathcal{I}|_p TM_p = T_p - \mathcal{I}(p)$, $v \neq 0$, is not of the form $v = \lambda \mathcal{I}(p)$, $\lambda \in \boldsymbol{R}$. For otherwise $\lambda \neq 0$ and $-\lambda^{-1} v = -\mathcal{I}(p)$

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$\in d\mathcal{g}|_p\, TM_p$. So $-\mathcal{g}(p)=w-\mathcal{g}(p)$ for some $w\in T_p$ and then $w=0$ a contradiction. Combining these observations we conclude that $d(\mathcal{P}\circ\mathcal{g})|_p=d\mathcal{P}|_{\mathcal{g}(p)}\circ d\mathcal{g}|_p$ is 1-1 and thus an isomorphism onto $TS^n_{\mathcal{P}(\mathcal{g}(p))}$ for each $p\in M$. It follows from the implicit function theorem that for each $p\in M$, $\mathcal{P}\circ\mathcal{g}$ maps some open neighborhood of p diffeomorphically onto an open neighborhood of $(\mathcal{P}\circ\mathcal{g})(p)\in S^n$. Hence for each $q\in S^n$, $(\mathcal{P}\circ\mathcal{g})^{-1}(q)$ is a discrete space. But $(\mathcal{P}\circ\mathcal{g})^{-1}(q)$ is a closed subspace of the compact space M and thus $(\mathcal{P}\circ\mathcal{g})^{-1}(q)$ is compact; it follows that $(\mathcal{P}\circ\mathcal{g})^{-1}(q)$ must be a finite set. Let $(\mathcal{P}\circ\mathcal{g})^{-1}(q)=\{p_1,\cdots,p_N\}$ with $p_i\neq p_j$ for $i\neq j$. From above, we know that for each i, $1\leqq i\leqq N$, there is an open neighborhood U_i of p_i which is mapped diffeomorphically onto an open neighborhood V_i of q. Since M is Hausdorff the U_i can be chosen disjoint. Then

$$V=\bigcap_{i=1}^{N} V_i\cap\left(S^n-(\mathcal{P}\circ\mathcal{g})\left(M-\bigcup_{i=1}^{N}U_i\right)\right)$$

is an open neighborhood of q such that $(\mathcal{P}\circ\mathcal{g})^{-1}(V)$ is a disjoint union of open sets each one of which is mapped diffeomorphically (and thus homeomorphically) onto V. It follows that $(\mathcal{P}\circ\mathcal{g})(M)$ is an open subset of S^n. But since M is compact, $(\mathcal{P}\circ\mathcal{g})(M)$ is also compact and thus closed and because S^n is connected we must have $(\mathcal{P}\circ\mathcal{g})(M)=S^n$. This shows that $\mathcal{P}\circ\mathcal{g}: M\to S^n$ is a covering map and because M is connected and S^n $(n\geqq 2)$ is simply connected, $\mathcal{P}\circ\mathcal{g}$ must be a homeomorphism; hence 1-1 and hence a diffeomorphism. This establishes assertion (1).

Since $\mathcal{P}\circ\mathcal{g}$ is 1-1, $\mathcal{g}$ must be 1-1. Hence $\mathcal{g}$ is an embedding (M is compact) and assertion (2) is established.

Consider the set $V=\bigcup_{p\in M}[0,\mathcal{g}(p))$. Clearly V is starshaped with 0 as a star-center. Set $W=\bigcup_{p\in M}\{t\mathcal{g}(p)\,|\,t>1\}$. Since $\mathcal{P}\circ\mathcal{g}$ is a homeomorphism it follows that W is connected and $V\cup W=\boldsymbol{R}^{n+1}-\mathcal{g}(M)$.

By the Jordan-Brouwer separation theorem $\boldsymbol{R}^{n+1}-\mathcal{g}(M)$ consists of two open components, one bounded X, and one unbounded Y, such that bdry $X=\mathcal{g}(M)=$ bdry Y. Since V and W are connected, disjoint, $V\cup W=\boldsymbol{R}^{n+1}-\mathcal{g}(M)$, and W is unbounded, it follows that $V=X$ and $W=Y$. Hence V is open and bdry $V=\mathcal{g}(M)$.

We will now show that V is the only open starshaped set whose point-set boundary equals $\mathcal{g}(M)$. Suppose V' is an open starshaped set such that bdry $V'=\mathcal{g}(M)$. Since V' is starshaped it is connected and hence either $V'\subset V$ or $V'\subset W$. Now using the easily established facts that V' and $\boldsymbol{R}^{n+1}-\mathrm{cl}\ V'$ are open, $V'\cup(\boldsymbol{R}^{n+1}-\mathrm{cl}\ V')=\boldsymbol{R}^{n+1}$

$-\mathcal{g}(M)$, $\boldsymbol{R}^{n+1}-\text{cl } V'$ is unbounded, V and W are the connected components of $\boldsymbol{R}^{n+1}-\mathcal{g}(M)$, it follows that $V'=V$.

We have seen that an arbitrary point of $Z=\boldsymbol{R}^{n+1}-\bigcup_{p\in M} T_p$ (which we took to be 0 by shifting the origin if necessary) is a star-center for V, i.e. is in kernel V. Hence $Z\subset$kernel V. We will establish $Z\subset$int(kernel V) by showing that Z is open. To show this we merely have to show that an arbitrary point of Z (which we again take to be 0) is in an open set N contained in Z, $0\in N\subset Z$. Consider the map $k: TM\to\boldsymbol{R}^{n+1}$ given by

$$k(v) = \mathcal{g}(\pi(v)) + d\mathcal{g}\big|_{\pi(v)}(v)$$

for all $v\in TM$ where $TM\equiv$the tangent bundle of M and $\pi: TM\to M$ is the canonical projection. Then $Z = \boldsymbol{R}^{n+1} - k(TM)$. Set $l=\sup_{p\in M}\|\mathcal{g}(p)\|$ which is finite since M is compact. Because k is continuous $k^{-1}(B_l)$ is closed where $B_l=\{x\in\boldsymbol{R}^{n+1}\,|\,\|x\|\leqq l\}$. We may pull back a Riemannian metric via $\mathcal{g}$, i.e. we can define an inner product for all pairs of vectors v, w such that $\pi(v)=\pi(w)$ by setting $(v, w)=(d\mathcal{g}|_{\pi(v)}(v),\ d\mathcal{g}|_{\pi(v)}(w))$ and this inner product is clearly continuous. Set $\|v\|=(v, v)^{1/2}$ for each $v\in TM$. Note that if $\|v\|>2l$ then

$$\begin{aligned}\|k(v)\| &= \|\mathcal{g}(\pi(v)) + d\mathcal{g}|_{\pi(v)}(v)\| \\ &\geqq \|d\mathcal{g}|_{\pi(v)}(v)\| - \|\mathcal{g}(\pi(v))\| \\ &\geqq \|v\| - l > 2l - l = l.\end{aligned}$$

This shows that $k^{-1}(B_l)\subset\{v\in TM\,|\,\|v\|\leqq 2l\}=Q$. But Q is compact since M is compact and the Tychonoff theorem is easily seen to carry over to this bundle situation. Hence, because $k^{-1}(B_l)$ is closed it is also compact. Thus $k(k^{-1}(B_l))=B_l\cap k(TM)$ is compact and hence closed. Finally

$$\begin{aligned}0 \in N &\equiv \text{int } B_l - k(TM) \\ &= \text{int } B_l - B_l \cap k(TM) \subset \boldsymbol{R}^{n+1} - k(TM) = Z\end{aligned}$$

and $N=\text{int } B_l-B_l\cap k(TM)$ is clearly open. Hence Z is open and $Z\subset$int(kernel V) as we wished to show.

Next we will establish the converse inclusion $Z\supset$int(kernel V). Consider an arbitrary point of int(kernel V) which we may take to be 0. Next take an arbitrary point p of M.

It follows from V being starshaped that cl V is also starshaped. It is also easy to show that kernel(cl V)$\supset$kernel V. Since $0\in$int(kernel V) there is an $\epsilon>0$ such $D_\epsilon=\{x\in\boldsymbol{R}^{n+1}\,|\,\|x\|<\epsilon\}$ $\subset$kernel V. Then $C=\bigcup_{\|x\|<\epsilon}[x, \mathcal{g}(p))\subset$cl V. Actually $C\subset V$. To see this, note first that $C=\bigcup_{0<\alpha\leq 1}\alpha(D_\epsilon)+(1-\alpha)\mathcal{g}(p)$ and is thus open. Secondly, it is easily seen that int(cl V) = V. So we have $C\subset$int(cl V) $= V$ as claimed.

Now suppose $0\notin Z$. Then $-\mathcal{g}(p)\in d\mathcal{g}|_p(TM_p)$ and so there is a curve γ in M such that $\gamma(0)=p$ and the tangent to $\mathcal{g}\circ\gamma$ at 0 is $-\mathcal{g}(p)$. It then follows that $(\mathcal{g}\circ\gamma)(t)\in C$ for sufficiently small positive t. But this contradicts $C\subset V$ and $\mathcal{g}(M)=$ bdry $V\subset \boldsymbol{R}^{n+1}-V$. Hence we must have $0\in Z$ as we desired. Therefore int(kernel V) $\subset Z$ and so int(kernel V) $=Z$ is established.

Finally we will establish the converse assertion. Suppose $\mathcal{g}(M)$ $=$ bdry V for some open starshaped set $V\subset \boldsymbol{R}^{n+1}$ such that int(kernel V) $\neq\emptyset$. We wish to show that $Z\neq\emptyset$.

Without loss of generality we may assume $0\in$int(kernel V). It is now sufficient to show that $0\notin d\mathcal{g}|_p(TM_p)$ for each $p\in M$. Take a $p\in M$. Since $0\in$ int(kernel V) there is an $\epsilon>0$ such that $\{x\in \boldsymbol{R}^{n+1} \mid \|x\|<\epsilon\}\subset$kernel V. Consider the set $C=\bigcup_{\|x\|<\epsilon}[x, \mathcal{g}(p))$. We will show that $C\subset V$.

Let $y=(1-\alpha)x+\alpha\mathcal{g}(p)$, $0\leqq\alpha<1$, $\|x\|<\epsilon$, be an arbitrary point of C. Since $\mathcal{g}(p)\in$ bdry V there is a sequence x_m in V such that $x_m\to\mathcal{g}(p)$. The sequence $y_m=(1-\alpha)^{-1}(y-\alpha x_m)=x+(1-\alpha)^{-1}\alpha(\mathcal{g}(p)-x_m)\to x$ and hence $\|y_m\|<\epsilon$ for some m. Then $y_m\in$kernel V and so $y=(1-\alpha)y_m+\alpha x_m\in V$. Hence $C\subset V$ as we wished to show.

Now the "curve argument" in the last paragraph of the proof of int(kernel V) $=Z$ can be used here and it gives the desired conclusion, $0\notin d\mathcal{g}|_p(TM_p)$. Hence $0\in Z\neq\emptyset$. This completes the proof of the theorem.

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Indiana University, Bloomington, Indiana 47401