MAPPINGS OF INDECOMPOSABLE CONTINUA

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Abstract. We show that every compact metric continuum is a continuous image of some indecomposable compact metric continuum.

Herein a continuum denotes a compact connected metric space. J. W. Rogers, Jr. has raised the questions: 'Which continua are continuous images of indecomposable continua?' and 'Is there an indecomposable continuum of which every indecomposable continuum is a continuous image?' [4]. We answer these questions.

Lemma 1. Every continuum irreducible between two points is a continuous image of some indecomposable continuum.

Proof. Let $M$ be a continuum irreducible between two points, $a$ and $b$. Let $D$ denote the well-known indecomposable continuum which is the union of all semicircles in the closed upper half-plane with center $(1/2, 0)$ and with both endpoints in the Cantor ternary set on the $x$-axis, and all semicircles in the lower half-plane with center $((1/2)((1/3)^n + (2/3)(1/3)^n), 0)$ for some nonnegative integer $n$ and with both endpoints lying in the Cantor set and on or to the left of the line $x = (1/3)^n$. (A sketch can be found in [3, p. 206], [2, p. 332], or [4, p. 96], so none is included here.)

Now, let

$$D_a = \{(x, y) \in D : x \leq 2/5\}, \quad D_b = \{(x, y) \in D : x \geq 3/5\},$$

$$A = \{y : (2/5, y) \in D_a\}$$

and observe that also

$$A = \{y : (3/5, y) \in D_b\}.$$

The continuum $D_M$ is now obtained from the disjoint union of $D_a, D_b$, and $M \times A$ by identifying $(2/5, y) \in D_a$ with $(a, y) \in M \times A$ and identifying $(3/5, y) \in D_b$ with $(b, y) \in M \times A$, for each $y \in A$. (Intuitively speaking we have removed from $D$ a copy of a closed interval crossed with $A$ and spliced in a copy of $M \times A$.) The proof that $D_M$ is indecomposable is straightforward, and is left to the reader. It de-
pends upon the indecomposability of $D$ and the irreducibility of $M$.

Now, we define $g : D_M \rightarrow M$ by

\[
\begin{align*}
g(p) &= a & \text{for } p \in D_a, \\
g(p) &= b & \text{for } p \in D_b, \\
g(m, y) &= m & \text{for } (m, y) \in M \times A.
\end{align*}
\]

This is the desired mapping of $D_M$ onto $M$.

**Theorem 2.** If $S$ is a continuum, there exists an indecomposable continuum of which $S$ is a continuous image.

**Proof.** Lemma 2 of [1] states that there is a continuous mapping of some continuum $M$ irreducible between two points onto $S$; and by the above lemma there is a continuous surjection $f : D_M \rightarrow M$. The composition of these maps yields a continuous surjection $f : D_M \rightarrow S$. Since $D_M$ is indecomposable, this is the desired mapping.

**Corollary 3.** There is no continuum of which every indecomposable continuum is a continuous image.

**Proof.** By the above theorem, every continuum would be a continuous image of such a continuum, contradicting the principal result of [5].

**Corollary 4.** If $S$ is a continuum there exists an indecomposable continuum $X$ which contains a copy of $S$ as a retract.

**Proof.** First, the mapping of $D_M$ onto $M$ can be considered a retraction by identifying $M$ with $M \times \{y\}$ for some $y \in A$. In turn, the mapping in Lemma 2 of [1] can be considered a retraction in a similar manner. Since composition of the retractions $g : D_M \rightarrow M$ and $r : M \rightarrow S$ yields a retraction, we set $X = D_M$ and the proof is complete.

**References**


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