CAN A 2-COHERENT PEANO CONTINUUM SEPARATE $E^3$?

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Abstract. The fact that there are unicoherent continua which separate $E^3$ is well known, e.g., a circle with a spiral converging onto it is such a continuum. In this paper we extend this pathology by describing a Peano continuum which separates $E^3$ and has the property that however it is written as the union of two unicoherent Peano continua, their intersection is unicoherent.

An inductive definition of $n$-coherence has been given by Transue in [5] in such a way that $0$-coherence is connectedness and $1$-coherence is unicoherence plus local connectedness. A unicoherent, locally unicoherent (i.e., having a basis of unicoherent regions) set $X$ is $2$-coherent provided that however $X$ is expressed as the union of two closed, locally connected, and unicoherent subsets $A$ and $B$, the set $A \cap B$ is unicoherent.

Interesting results are obtained for $2$-coherent sets: any unicoherent, locally unicoherent Peano continuum in $E^3$ which is not $2$-coherent must separate $E^3$. Among the conjectures proposed by Transue are the following special cases:

**Conjecture 1.** A $2$-coherent, unicoherent continuum in $E^3$ does not separate $E^3$.

**Conjecture 2.** Any retract of a $2$-coherent space is itself $2$-coherent.

In the introduction of [5], the author suggests that it is not known if the assumption of local unicoherence adds anything to the definition of $2$-coherence. We show by an example that it does make a difference, and that the above conjectures are false without it. Let a space be called "$2$-coherent in the wide sense" if it satisfies the definition of $2$-coherence except that the requirement of local unicoherence is omitted.

Consider the cylindrical shell $C = S^1 \times [0, 1] \times [0, 1]$ in $E^3$, where the middle factor refers to the altitude of $C$ and the third factor to the thickness of $C$. Let $C_n = S^1 \times [0, 1] \times [1/2^{n+1}, 1/2^n]$, and remove from $C_n$ an open solid rod $R_n$ of diameter $1/2^{n+1}$, which is tangent to...
the helical line $L_n$ where $L_n = \{(e^{2\pi n t}, t, 1/2^n) : 0 < t \leq 1, n \text{ even and } 0 \leq t < 1, n \text{ odd}\}$. The boundary of $R_n$ in $C_n$ is a tube $T_n$ which is capped at the bottom or top, depending on whether $n$ is even or odd. If $Y = C \setminus \bigcup_{n=1}^{\infty} R_n$ and the two sets $S^1 \times \{0\} \times \{0\}$ and $S^1 \times \{1\} \times \{0\}$ are each identified to a point, a quotient space $X$ is obtained from $Y$, and $X$ is the desired space. We carry out this identification in $E^3$ by removing the denoted rods from $C$, and then pinching its top and bottom annuli so that their inner circular boundaries are shrunk to points; thus $X$ is embedded in $E^3$.

Let us call the quotient map $q$, and note that $q(S^1 \times [0, 1] \times \{0\}) = S$ is a 2-sphere. Moreover, $S$ bounds one of the two complementary domains of $X$ in $E^3$, and there is a retraction (the projection) of $X$ onto $S$. Thus $X$, which we shall show to be 2-coherent in the wide sense, separates $E^3$ and allows a retraction onto the sphere $S$, which is not 2-coherent in either sense.

**Lemma 1.** If a unicoherent Peano continuum $X$ is not 2-coherent in the wide sense, then there is an essential map $F: X \rightarrow S^2$.

**Proof.** If $X = A \cup B$, with $A$ and $B$ unicoherent Peano continua and $A \cap B$ is not unicoherent, then there is an essential map $f: A \cap B \rightarrow S^1$ [6, Chapter 8]. Consider $S^1$ as the equator of $S^2$ with $N$ and $Z$ the northern and southern hemispheres, respectively, of $S^2$. Then $f$ has an extension $F_A: A \rightarrow N$ and an extension $F_B: B \rightarrow Z$ by the Tietze extension theorem. Then if $F = F_A \cup F_B$, $F$ maps the proper triad $(X; A, B)$ into $(S^2; N, Z)$ and hence the following commutative diagram, with exact rows, exists: (Čech cohomology, integral coefficients)

$$
\begin{array}{cccccc}
0 &=& H^1(A) &+& H^1(B) &\rightarrow& H^1(A \cap B) &\rightarrow& H^2(X) \\
& & f^* \uparrow & & F^* \uparrow & & \\
0 &=& H^1(N) &+& H^1(Z) &\rightarrow& H^1(S^1) &\rightarrow& H^2(S^2) \\
\end{array}
$$

Since $f$ is essential, $f^*$ is nonzero, and $\delta$ is 1-1 so therefore $F^*$ is nonzero and $F$ must be essential. Lemma 1 was proven by Transue in [5] and is included here for completeness.

**Theorem 1.** $X$ is 2-coherent in the wide sense, and is a unicoherent Peano continuum.

**Proof.** $X$ is clearly compact and connected. Moreover, $X$ is locally connected since it has a basis of connected open sets in the relative topology. If $X$ were not unicoherent, there would be a simple closed curve $J \subset X$ which is a retract of $X$, and hence the
infinite cyclic group $H_1(J)$ would be a subgroup of $H_1(X)$. We show that $H_1(X) = 0$ (Čech homology, integral coefficients) to establish that $X$ is unicoherent. Setting $P_n = q(C \setminus \bigcup_{k=1}^{n} R_k)$, we see that $\bigcap_{n=1}^{\infty} P_n = X$. By the continuity of Čech homology, $H_1(X) = \lim_{n\to\infty} H_1(P_n) = 0$ since each $P_n$ is a strong deformation retract of a spherical shell $\approx S^2 \times [0, 1]$. 

Now suppose that $X$ were not 2-coherent in the wide sense; then $X = A \cup B$ with $A$ and $B$ unicoherent Peano continua and $A \cap B$ not unicoherent. We can construct the essential map $F: X \to S^2$ which was described in Lemma 1. Then $F$ cannot be extended to a three cell containing $X$ [1, p. 347] and so the homomorphism $F_*: H_2(X) \to H_2(S^2)$ is nonzero [4, p. 147].

We next show that the restriction of $F$ to $S$ maps $S$ onto $S^2$. Let the set $U_n = X \setminus \bigcup_{k=1}^{n} q(C_k \setminus R_k)$; $U_n$ is a neighborhood of $S$ in $X$. Observe that there is a deformation retraction $r: X \to U_n$ obtained by squashing $C_1 \setminus R_1$ into $C_1 \cap C_5$, and then $C_2 \setminus R_2$ into $C_5 \cap C_9$, and so on, a finite number of times. Thus we have a commutative diagram

\[
\begin{array}{ccc}
H_2(X) & \xrightarrow{F_*} & H_2(S^2) \\
\downarrow{r_*} & & \downarrow{[F| \bigcup_{n}]_*} \\
H_2(U_n) & & 
\end{array}
\]

and since $F_*$ is nonzero, $[F| \bigcup_{n}]_*$ must also be nonzero. Hence $F|\bigcup_{n}: \bigcup_{n} \to S^2$ is an onto map for every integer $n$, so $F| S: S \to S^2$ is onto. Recalling the construction of $F$ in Lemma 1, we see that neither $A$ nor $B$ can contain $S$, or else $S$ would be mapped into one hemisphere of $S^2$.

Let $p$ be a point of $S$ which is not in $A$, and which is different from both the points $q(S^1 \times \{1\} \times \{0\})$ and $q(S^1 \times \{0\} \times \{0\})$; we call these points the “north” and “south” poles of $X$, respectively. Let $R$ be a region about $p$ which does not meet $A$. By the method of construction for $X$, for some integer $N$ and all $n > N$, $R$ must contain a “cross section” of every tube $q(T_n)$. Since $B$ is unicoherent and contains $R$, $B$ must contain that portion of each even numbered tube between $R$ and the north pole of $X$, for $n > N$. Otherwise $B$ could be retracted onto a simple closed curve in a tube $T_n$, contradicting the fact that each retract of a unicoherent Peano continuum is itself unicoherent [6, Chapter 8]. Similarly, $B$ contains that portion of each odd numbered tube between $R$ and the south pole of $X$, for $n > N$. However, the union of these portions of tubes mentioned is dense in $S = q(S^1 \times [0, 1] \times \{0\})$, and hence $B$ contains $S$, a contradiction.
Conjecture 1 remains an interesting open problem. Transue has given an affirmative answer for polyhedra in [5]. The example in this paper indicates that the technique of approximation by polyhedra may not be a useful way to attack Conjecture 1 unless local unicoherence is somehow utilized. The author has extended Transue’s elegant proof that polyhedra obey Conjecture 1. Suppose that \( X \subseteq E^2 \) is a unicoherent Peano continuum, \( E^3 \setminus X \) has components \( A \) and \( B \), and there is a set \( T \subseteq X \) such that \( T = h(D \times [a, b]) \), \( h \) a homeomorphism, \( D \) a closed 2-cell, and \( a, b \) real numbers. If, in addition, \( \text{Fr}(A) \cap T = h(D \times \{a\}) \), \( \text{Fr}(B) \cap T = h(D \times \{b\}) \), and \( h(D \times (a, b)) \) is contained in the interior of \( X \), then \( X \) is not 2-coherent even if \( a \neq b \).

References