BAIRE FUNCTIONS AND THEIR RESTRICTIONS
TO SPECIAL SETS

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Abstract. A function $f$ from a complete and separable metric space $X$ into the real numbers is of Baire class 1 iff for every nonempty perfect subset $H$ of $X$, $f|H$ contains a point where $f|H$ is continuous. This paper examines a similar idea obtained by changing "perfect subset $H$" to "union of a countable collection of perfect subsets" in the preceding characterization of Baire class 1 functions. This new idea is also characterized by using "condensation points" and "totally imperfect sets." Functions of this new type are of Baire class 1. However, the converse is false.

Introduction. There are various characterizations for functions of Baire class 1 in different spaces. For example see [5, p. 279], [6] and [7]. In a separable and complete metric space $X$, a function of Baire class 1 can be characterized as a function $f:X\to\mathbb{R}$ ($\mathbb{R}$ denotes the real numbers) such that if $H$ is a nonempty perfect subset of $X$, then $f|H$ (the restriction of $f$ to the domain $H$) contains a point where $f|H$ is continuous [4, p. 289]. Any function described as of Baire class 1 in this paper will mean a function that satisfies the preceding characterization. This paper examines an alteration of the preceding characterization of Baire class 1 and its relation to Baire functions. The alteration is stated in the following definition.

Definition 1. A function $f$ from a separable and complete metric space $X$ into the real numbers has property P means that if $H$ is the nonempty union of a countable collection of perfect subsets of $X$, then $f|H$ contains a point where $f|H$ is continuous.

The first theorem establishes two other characterizations of property P. There exist functions of Baire class 1 for which this new condition (property P) is false; the paper shall conclude with an example of such a function. The uniform limit of a sequence of Baire class 1 functions is a function of Baire class 1 [5]. However, the uniform limit of a sequence of functions, each with property P, does not necessarily have property P. Equivalent characterizations of property P using condensation points and totally imperfect sets are derived (a totally imperfect set is a set that contains no perfect set [4, p. 201]).
Theorem 1. Suppose $X$ is a separable and complete metric space. Suppose $f$ is a function from $X$ into $R$. The following statements are equivalent:

(a) If $H$ is the nonempty union of a countable collection of perfect subsets of $X$, then $f|H$ contains a point where $f|H$ is continuous.

(b) If $H$ is a nonempty subset of $X$ such that every point of $H$ is a condensation point of $H$, then $f|H$ contains a point where $f|H$ is continuous.

(c) If $H$ is a nonempty totally imperfect subset of $X$ and every point of $H$ is a condensation point of $H$, then $f|H$ contains a point where $f|H$ is continuous.

Definition 2. The function $f:X\rightarrow R$ is a step-like function means that if $B$ is a nonempty subset of $X$, then there is an open set $U$ containing a point of $B$ such that $f|B\cap U$ is constant.

Theorem 2. Suppose $X$ is a separable and complete metric space. Suppose $f$ is a function from $X$ into $R$. The following statements are equivalent:

(a) $f$ is of Baire class 1.

(b) $f$ is the uniform limit of a sequence of step-like functions.

(c) $f$ is the uniform limit of a sequence of functions each with property $P$.

Lemma. If $F$ is a totally imperfect subset of a complete, separable and perfect metric space $X$ and $E$ is a countable subset of $X$, then $F+E$ is totally imperfect.

1. Proof of Lemma. Suppose $F+E$ is not totally imperfect. Let $D$ denote a nonempty perfect subset of $F+E$. Therefore $D-E$ is a subset of $F$ and an uncountable Borel subset of $X$. Consequently [1], [3], [4, p. 205], $D-E$ contains a homeomorphic image of the Cantor set. Contradiction. Therefore $F+E$ is totally imperfect.

2. Proof of Theorem 1. Proof that (a) implies (b). Suppose (a). Suppose $H$ is a nonempty subset of $X$ such that every point of $H$ is a condensation point of $H$ and $f|H$ is discontinuous at each of its points. For each positive number $c$ let $H_c$ denote the subset of $X$ such that $x \in H_c$ iff the saltus of $f|H + \{(x, f(x))\}$ at $(x, f(x))$ is greater than or equal to $c$. For each $c$, $H_c$ is closed. There exists a positive number $d$ such that $H_d$ is uncountable and if $c$ is a positive number less than $d$, then $H_c$ is uncountable. For each positive number $c \leq d$, there exists an uncountable and perfect set $K$ and a nonempty countable set $M$ such that $H_c = K_c + M$ [4, p. 159]. Let $M$ denote the union of the sets $M_{a/1}, M_{a/2}, M_{a/3}, \cdots$. Let $P_1, P_2, P_3, \cdots$ denote a sequence of all
the points of $M$. $N(P, e)$ will denote the neighborhood of the point $P$ with radius $e$. For each positive integer $m$ and each positive integer $n$, $HN(P_n, \frac{1}{m})$ is uncountable. Therefore there is a positive integer $i$ such that $\text{H}_{d_i}\text{Cl}(N(P_n, \frac{1}{m}))$ is uncountable and closed. Therefore there is a nonempty perfect subset of $\text{H}_{d_i}\text{Cl}(N(P_n, \frac{1}{m})) \subseteq \text{H}_{d_i}N(P_n, 1/m)$. Let $L_n(m)$ denote one such set for each $m$ and $n$. For each $n$, let $L_n$ denote the union of $L_n(1), L_n(2), L_n(3), \cdots$ and $\{P_n\}$. $L_n$ is a nonempty perfect set.

Let $G$ denote the countable collection of nonempty perfect subsets of $X$ such that $C \in G$ iff there is a positive integer $n$ such that $C = \text{K}_{d/n}$ or $C = L_n$. $G^*$ (the union of the elements of $G$) is the nonempty union of a countable collection of perfect subsets of $X$. Therefore $f|G^*$ contains a point $(y, f(y))$ where $f|G^*$ is continuous. Therefore there is an $n$ such that $y \in \text{H}_{d/n}$. Therefore $f|H + \{(y, f(y))\}$ is not continuous at $(y, f(y))$. $H$ is a subset of $G^*$. Therefore $f|G^*$ is not continuous at $(y, f(y))$. Contradiction. Therefore (a) implies (b).

Proof that (b) implies (c). Trivial.

Proof that (c) implies (a). Suppose (c). Suppose there is a nonempty countable collection $G$ of nonempty perfect subsets of $X$ such that $f|G^*$ contains no point where $f|G^*$ is continuous. Let $K$ denote the union of all perfect subsets of $X$. Therefore $K$ is an uncountable, perfect, separable and complete subspace of $X$. Therefore $K$ is the sum of two disjoint, and totally imperfect sets $L$ and $M$ [4, pp. 201–202]. Every point of $L$ is a condensation point of $L$. For suppose $P$ is a point of $L$ that is not a condensation point of $L$. There are open sets $U$ and $V$ each containing $P$ such that $UL$ is countable and $\text{Cl} V \subseteq U$. $(\text{Cl} V)K$ is closed and uncountable. Therefore $(\text{Cl} V)K$ contains a nonempty perfect set. But $(\text{Cl} V)K \subseteq (UL) + M$. Therefore $(UL) + M$ contains a nonempty perfect subset. Therefore, by the Lemma, $M$ is not totally imperfect. Contradiction. Likewise, every point of $L$ is a condensation point of $M$. Likewise, every point of $M$ is a condensation point of $M$ and of $L$. Similarly, it can be shown that every point of $LG^*$ is a condensation point of $LG^*$ and of $MG^*$ and also that every point of $MG^*$ is a condensation point of $MG^*$ and of $LG^*$. $LG^*$ and $NG^*$ are two disjoint, uncountable and totally imperfect sets such that $G^* = LG^* + MG^*$.

In the following argument, a function and its graph are regarded as one and the same. $K \times R$ is separable since $K$ and $R$ are both separable metric spaces. $f|G^*$ is a subset of $K \times R$. Therefore $f|G^*$ is separable [2, pp. 141–142]. Let $E$ denote a countable subset of $G^*$ such that $f|E$ is a countable and dense subset of $f|G^*$. By the Lemma, $LG^* + E$ is a totally imperfect subset of $X$. Also, every point of $LG^*$
+E is a condensation point of LG*. Therefore \( f' \) (LG* +E) contains a point \( P \) where \( f \) (LG* +E) is continuous. But \( f \) | G* is also continuous at \( P \) because \( f \) | E is dense in \( f \) | G*. Contradiction. Therefore (a). Therefore (c) implies (a).

Therefore (a), (b) and (c) are equivalent.

3. **Proof of Theorem 2.** Proof that (a) implies (b). Suppose \( f \) is of Baire class 1. Suppose the collection \( T \) of open sets of \( X \) is well-ordered by “precedes”. If \( H \) is a subcollection of \( T \) and \( U \) is an open set, then \( I[H, U] \) will denote \( U \) minus the union of the open sets in \( H \) that precede the open set \( U \).

**Definition 3.** For each positive integer \( n \), a subcollection \( H \) of \( T \) is of type \( A(n) \) means that \( H \) contains the first element \( U \) of \( T \) that has the property that if \( x \) and \( y \) are in \( U \), then \( |f(x) - f(y)| < 1/n \) and if \( V \) is in \( H \), then \( V \) is the first element of \( T \) that has the property that if \( x \) and \( y \) are in \( I[H, V] \), then \( |f(x) - f(y)| < 1/n \) and \( I[H, V] \) is not empty.

For each positive integer \( n \), let \( K(n) \) denote the union of all the sets of type \( A(n) \). \( K(n) \) is of type \( A(n) \). \( K(n) \) covers \( X \). For each \( U \) in \( K(n) \) let \( a(U) \) denote a point of \( I[K(n), U] \) and \( f_n | I[K(n), U] \) denote the constant function, \( f_n : I[K(n), U] \rightarrow \{ f(a(U)) \} \). Let \( f_n \) denote the function from \( X \) into \( R \) such that if \( U \) is an element of \( K(n) \), then \( f_n | I[K(n), U] \) is a subset of \( f_n \). Suppose \( B \) is a nonempty subset of \( X \). Let \( U(B) \) denote the first element of \( K(n) \) that contains a point of \( B \). Let \( b \) denote a point of \( BU(B) \). \( b \) is an element of \( I[K(n), U(B)] \). If \( x \) is an element of \( BU(B) \), then \( x \) is in \( I[K(n), U(B)] \) and \( f_n(b) = f_n(x) = f(a(U)) \). Therefore \( f_n | B \) is constant over \( BU(B) \). Therefore \( f_n \) is a step-like function.

If \( c > 0 \), there is a positive integer \( N \) such that \( 1/N < c \) and if \( n \geq N \) and \( x \in X \), then \( 1/n \leq 1/N < c \) and there is a first open set \( U \) in \( K(n) \) that contains \( x \). Therefore \( f_n(x) = f(a(U)) \). Since \( a(U) \) and \( x \) are both in \( I[K(n), U] \), \( |f(a(U)) - f(x)| < 1/n < c \). Therefore \( |f_n(x) - f(x)| < c \). Therefore \( f_n \) converges uniformly to \( f \). Therefore (a) implies (b).

Proof that (b) implies (c). Since every step-like function has property P, (b) implies (c) directly.

Proof that (c) implies (a). Suppose (c). Let \( f_1, f_2, f_3, \ldots \) denote a sequence of functions, each with property P, that converges uniformly to \( f \). If \( H \) is a nonempty perfect subset of \( X \), then \( H \) is the union of a countable collection of nonempty perfect subsets of \( X \). Therefore, for each positive integer \( n \), \( f_n | H \) contains a point where \( f_n | H \) is continuous. Therefore \( f_n \) is of Baire class 1. Therefore \( f \) is the uniform limit of a sequence of functions each of Baire class 1. Therefore \( f \) is of
Baire class 1 \[4, p. 269\]. Therefore (c) implies (a). There (a), (b) and (c) are equivalent.

4. Example. For each positive integer \(n\) and each positive integer \(k\) such that \(1 \leq k \leq 3^{n-1}\), let \(c(n, k)\) denote the Cantor set over the interval \([ (3k-2)3^{-n}, (3k-1)3^{-n} ]\). For each positive integer \(n\), let \(f_n\) denote the function from \([0, 1]\) into \([0, 1]\) such that if \(x\) is in \(c(n, k)\) \(-\) \([ (3k-2)3^{-n}, (3k-1)3^{-n} ]\), then \(f_n(x) = 1/n\). Otherwise \(f_n(x) = 0\).

Let \(c(0)\) denote the Cantor set over the interval \([0, 1]\). Let \(f_0\) denote the function from \([0, 1]\) into \([0, 1]\) such that if \(x\) is in \(c(0)\), then \(f_0(x) = 1\). Otherwise \(f_0(x) = 0\).

Let \(f = \sum f_n\). \(f\) is of Baire class 1. Let \(C\) denote the subset of \([0, 1]\) such that \(x \in C\) iff \(f(x) \neq 0\). \(C\) is the nonempty union of a countable collection of perfect sets. \(f\) \(\big|\) \(C\) contains no point where \(f\) \(\big|\) \(C\) is continuous. Therefore \(f\) does not have property P. Therefore property P and Baire class 1 are not equivalent.

References


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