SEMICONTINUITY OF NULLITY OR DEFICIENCY IMPLIES NORMABILITY OF THE SPACE

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Abstract. In this paper the upper semicontinuity of nullity and deficiency on locally convex spaces is examined. If either is semicontinuous in the topology of uniform convergence on bounded sets on $L(X)$, then $X$ is normable. If the invertible elements in $L(X)$ are open, then $X$ is normable. The results are applied to topological algebras.

Let $X$ be a locally convex Hausdorff linear topological space. Let $L(X)$ be the space of all continuous linear maps from $X$ into $X$ with the topology of uniform convergence on bounded sets. If $T \in L(X)$, define $\tilde{T} : X/\ker T \to X$ by $\tilde{T}(x + \ker T) = Tx$. Let $X/\ker T$ have the quotient topology. There is a $T^{-1} : R(T) \to X/\ker T$. $T^{-1}$ is continuous if and only if (for a normed space) $\gamma(T) = \inf \{|Tx|/d(x, \ker T)|$, the (reduced) minimum modulus of $T$, satisfies $\gamma(T) > 0$, if and only if (for a Banach space) $R(T)$ is closed.

If $X$ is normable, then several results on the semicontinuity of the nullity $\alpha(T) = \dim ker T$ and the deficiency $\beta(T) = \text{codim cl}(R(T))$ are known, such as the following result due to Webb.

Proposition [6]. Let $X$ be a normed linear space and $T \in L(X)$ with $T^{-1}$ continuous. Then there is a neighborhood $N$ of $T$ in $L(X)$ such that, for any $A \in N$,

1. $\alpha(A) \leq \alpha(T)$,
2. $\beta(A) \leq \beta(T)$.

Corollary. Let $X$ be a normed linear space. There is a neighborhood $N$ of $I \in L(X)$ such that, for any $A \in N$,

1. $\alpha(A) = 0$,
2. $\beta(A) = 0$.

We shall show that if these conclusions hold then the space $X$ must be normable.

Proposition. Let $X$ be a l.c.s. If there is a neighborhood $N$ of $I \in L(X)$ for the topology of uniform convergence on bounded sets such that for $A \in N$ either $\alpha(A) = 0$ or $\beta(A) = 0$, then $X$ is normable.
Proof. There is a closed, convex, circled and bounded set $B$ and a convex circled neighborhood of zero $V$ such that if $A \in N(B,V) = \{ A \in L(X) : AB \subseteq V \}$ then either $\alpha(I - A) = 0$ or $\beta(I - A) = 0$.

Assume if possible that there is an element $x_0 \in V \setminus B$. Then $x_0 \neq 0$.

By the separation theorems [2] there is a continuous linear functional $m'$ on $X$ such that $|m'(b)| \leq |m'(x_0)| > 0$ for all $b \in B$. Set $m(x) = m'(x)/m'(x_0)$. Then $|m(b)| \leq 1$ for all $b \in B$ and $m(x_0) = 1$. Define $Ax = m(x)x_0$. $A \in L(X)$ and since $|m(b)| \leq 1$ and $V$ is circled, $Ab = m(b)x_0 \in V$ for all $b \in B$. Thus $A \in N(B,V)$. $(I - A)x_0 = 0$, so $\alpha(I - A) \neq 0$. If $(I - A)x = x_0$, then $x - m(x)x_0 = x_0$, $x = (1 + m(x))x_0$, and $(I - A)x = 0$. Thus $x_0 \in R(I - A)$. Since $R(I - A)$ is closed [3], $\beta(I - A) \neq 0$. Thus we must have $V \subseteq B$, so $V$ is a bounded neighborhood of zero, and $X$ is normable.

Corollary. $X$ is normable if and only if for every $T \in L(X)$ with $T^{-1}$ continuous there is a neighborhood $N$ of $T$ in $L(X)$ such that for $A \in N$ either

(1) $\alpha(A) \leq \alpha(T)$ or
(2) $\beta(A) \leq \beta(T)$.

Corollary. $X$ is normable if and only if the invertible elements in $L(X)$ are open in the topology of uniform convergence on bounded sets.

It should be noted that perturbations of $I$ may be invertible for operators in some subset of $L(X)$. For example, the following is a corollary of a result of Vladimirskii [5].

Proposition. Let $X$ be a complete l.c.s. Let $U$ be a closed absolutely convex neighborhood of zero. Then there is a closed absolutely convex neighborhood of zero $V$ such that if $\alpha \in L(X)$ is open, $\alpha(U)$ is bounded, and $\alpha(U) \subseteq V$ then $I + \alpha$ is invertible.

Results related to the above corollary have been obtained by Blair [1] and Williamson [7]. $X$ is normable if and only if there is a subalgebra $A$ of $L(X)$ containing all operators of finite rank such that multiplication $A \times A \rightarrow A$ is continuous for the topology of uniform convergence on bounded sets, compact sets, or pointwise convergence on $A$. $X$ is normable if and only if there is a linear set $L$ in $L(X)$ containing all operators of finite rank which can be given a linear topology such that $(T, x)\rightarrow Tx$ is continuous.

The following application to topological algebras (locally convex topological rings with identity) was suggested by this paper's referee. A topological algebra $X$ has a continuous inverse [4] if there is a neighborhood $W$ of the identity $e$ such that every element of $W$ has an
inverse and the mapping \( x \rightarrow x^{-1} \) is continuous on \( W \). In this case the invertible elements are open and \( x \rightarrow x^{-1} \) is continuous. The left regular representation of \( X \) is \( R: X \rightarrow L(X) \) defined by \( R(a)x = ax \).

**Proposition.** Let \( X \) be a topological algebra whose left regular representation contains the operators of finite rank. If \( X \) has a continuous inverse, then \( X \) is normable.

**Proof.** The map \( a \rightarrow R(a) \) gives \( R(X) \) the topology of pointwise convergence. There is a neighborhood \( U \) of \( I = R(e) \) for the stronger topology of uniform convergence on bounded sets such that \( U \cap R(X) \subseteq R(W) \), whose elements are invertible. Thus there is a closed, convex, circled and bounded set \( B \subseteq X \) and a convex circled neighborhood \( V \) of \( 0 \) such that \( I + N(B, V) \cap R(X) \subseteq R(W) \). If there is an \( x_0 \in V \setminus B \), then as before there is a continuous linear functional \( m \) such that \( |m(B)| \leq 1 \) and \( m(x_0) = 1 \). If \( Ax = m(x)x_0 \), then \( -A \in N(B, V) \cap R(X) \) and \( (I-A)x_0 = 0 \). Thus \( I-A \in R(W) \) is not invertible, and so \( V \subseteq B \) and \( X \) is normable. □

Applying the left regular representation to the results of Blair and Williamson \[7\], we obtain the following result.

**Proposition.** Let \( X \) be a topological algebra whose left regular representation contains the operators of finite rank. Then \( X \) is normable if and only if multiplication \( X \times X \rightarrow X \) is continuous.

**References**


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