ON THE ABSOLUTE CONTINUITY OF THE LIMIT RANDOM VARIABLE IN THE SUPERCRITICAL GALTON-WATSON BRANCHING PROCESS

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Abstract. Let \( \{Z_n: n\geq 0\} \) be a simple Galton-Watson branching process with offspring distribution \( \{p_j\} \) satisfying \( 1 < \sum j p_j < \infty \). It is known that there exist constants \( C_n \) such that \( W_n = Z_n C_n \) converges with probability one to a nondegenerate limit random variable \( W \). Here we show that this \( W \) is always absolutely continuous on \((0, \infty)\).

Let \( \{Z_n: n\geq 0\} \) be a Galton-Watson branching process with offspring probability generating function \( f(s) = \sum p_j s^j \). Assume \( P(Z_0 = 1) = 1 \), \( \{p_j\} \) is nondegenerate, \( 1 < m = \sum j p_j < \infty \) and \( p_0 = 0 \). Seneta [5] and Heyde [3] have shown that there always exists a sequence of constants \( C_n \rightarrow \infty \), \( C_n^{-1} C_{n+1} \rightarrow m \) such that \( W_n = Z_n C_n^{-1} \) converges with probability one to a nondegenerate random variable \( W \). Kesten and Stigum [4] had shown earlier that when \( \sum p_j j \log j < \infty \), \( C_n \) may be taken as \( m^n \) and in this case the limit \( W \) has an absolutely continuous distribution on \((0, \infty)\). Their main tool was to show that if \( \phi \) is the characteristic function of \( W \) then \( \phi' \), the derivative, is integrable. This will fail when \( \sum p_j j \log j = \infty \) since in this case \( EW = \infty \) and the existence of \( \phi' \) is not guaranteed, let alone its integrability. In this paper we shall present a simple idea to show that \( W \) is always absolutely continuous.

If \( \phi(it) = E(e^{itW}) \) is the characteristic function of \( W \) then it is easily seen using the fact \( C_n^{-1} C_{n+1} \rightarrow m \) that \( \phi \) satisfies the so-called Abel's functional equation.

\[
\phi(it) = f(\phi(it/m)).
\]

The equation and the nondegeneracy of \( W \) ensures that \( |\phi(it)| < 1 \) for \( t \neq 0 \). We exploit this to get a rate of decay for \( \phi(it) \) as \( t \rightarrow \infty \). First we recall the following about the rate of convergence of \( f_n(x) \) for \( |x| < 1 \).

Lemma 1. If \( p_1 > 0 \) then \( p_1^{-n} f_n(x) \uparrow Q(x) < \infty \) for \( 0 \leq x < 1 \). If \( p_1 = 0 \) then for any \( \epsilon > 0 \), \( \lim_{n \rightarrow \infty} \epsilon^{-n} f_n(x) = 0 \) for \( 0 \leq x \leq 1 \).
For a proof see [1].

Let \( 0 < \delta \leq \infty \) be defined by \( p_1 = m^{-\delta} \).

**Lemma 2.** If \( p_1 > 0 \) then \( \sup_\theta |t|^\theta |\phi(it)| < \infty \). If \( p_1 = 0 \) then, for any \( \theta > 0 \), \( \sup_\theta |t|^\theta |\phi(it)| < \infty \).

**Proof.** By continuity \( \beta = \sup_{s \in [0, 1]} |\phi(is)| < 1 \). Iterating (1) we get \( \phi(im^nt) = f_n(\phi(it)) \) and hence \( \sup_{s \in [0, 1]} |\phi(im^nt)| \leq f_n(\beta) \). Now use Lemma 1 to complete the proof. q.e.d.

If \( \delta > 1 \) then Lemma 2 says that \( \phi \) is integrable and so \( W \) is absolutely continuous on \( (0, \infty) \). In fact, it has a uniformly continuous density function. Assume for the rest of the paper that \( \delta \leq 1 \). Let \( k \) be the smallest integer such that \( k \delta > 1 \).

**Lemma 3.** For all \( r \geq k \), the \( r \) fold convolution \( S_r \) of \( W \) is absolutely continuous on \((0, \infty)\) and has a uniformly continuous density.

**Proof.** The characteristic function of \( S_k \) is \( \phi^k(it) \) and, by Lemma 2, \( \sup_\theta |t|^\theta |\phi^k(it)| < \infty \). But \( k \delta > 1 \) and so \( \phi^k \) and \( \phi^r \) for \( r \geq k \) are integrable. q.e.d.

The following is a key step.

**Lemma 4.** Let \( W_j, j = 0, 1, 2, \ldots, \) be independent random variables with the same distribution as \( W \). Assume further that the \( W_j \) sequence is independent of our Galton-Watson branching process \( \{Z_n\} \). Then \( W_0 \) has the same distribution as \( (1/m^n) \sum_{j=1}^r W_j \) for each \( n \).

**Proof.** The characteristic functions of the above two random variables are respectively \( \phi(it) \) and \( f_n(\phi(it/m^n)) \). They are equal for all \( t \) as can be seen by iterating (1) \( n \) times. q.e.d.

We shall now show that the absolute continuity of \( S_k \) for all large \( k \) implies the same for \( W \).

**Lemma 5.** Let \( E \) be a Borel set with Lebesgue measure zero. Then, \( P(W \in E) = 0 \).

**Proof.** From Lemma 4 we see that

\[
P(W_0 \in E) = \sum_{r=1}^{\infty} P(Z_n = r, \frac{1}{m^n} \sum_{i=1}^r W_j \in E)
\]

\[
= \sum_{r=1}^{\infty} P(Z_n = r, \frac{1}{m^n} \sum_{j=1}^r W_j \in E)
\]

\[
= \sum_{r=1}^{\infty} P(Z_n = r) P\left( \sum_{j=1}^r W_j \in m^n E \right)
\]

(by independence of \( \{Z_n\} \) and \( \{W_j\} \)).
But by Lemma 3, for \( r \geq k \), \( P(\sum_{i=1}^{r} W_i \leq m^k E) = 0 \). Thus \( P(W_0 \in E) \leq P(Z_n < k) \) for each \( n \). Clearly, \( P(Z_n < k) \to 0 \) as \( n \to \infty \) for each fixed \( k \). q.e.d.

Lemma 5 does not assert that the density \( w(x) \) of \( W \) is continuous.\(^1\)

If this is the case then the argument in [2] yields the conclusion that \( w(x) > 0 \) for all \( x > 0 \). One can perhaps then prove a local limit theorem for the \( \sum p_{ij} \log j = \infty \) case in analogy with the results of [2].

References


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\(^1\) Added in proof. S. Dubuc has just shown that \( w(x) \) is continuous by using different techniques. He has many more results on this topic. See S. Dubuc, Ann. Inst. Fourier (Grenoble) 21 (1971), 171–251.