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COMMUTATIVE RINGS IN WHICH EVERY PRIME IDEAL IS CONTAINED IN A UNIQUE MAXIMAL IDEAL

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ABSTRACT. The class of rings with 1 satisfying the properties of the title is characterized by some separation properties on the prime and maximal spectra, and, in such rings, the map which sends every prime ideal into the unique maximal ideal containing it, is continuous. These results are applied to $C(X)$ to obtain Stone's theorem and the Gelfand-Kolmogoroff theorem. As a side result, the methods give new information on the mapping $P \to P \cap C^*(X)$ (P a prime ideal of $C(X)$).

Introduction. A ring with an identity possessing the properties of the title will be called a pm-ring. The purpose of this paper is to study some properties of the prime spectrum $\mathcal{P}$ and of the maximal spectrum $\mathfrak{M}$ of a pm-ring A. The pm-property is strictly related to separation properties of $\mathcal{P}$ and $\mathfrak{M}$; specifically, it is equivalent to the normality of $\mathcal{P}$ and implies that $\mathfrak{M}$ is T_2, all three conditions being equivalent if the Jacobson radical and the nilradical of A coincide. In a pm-ring A, let μ be the map of $\mathcal{P}$ onto $\mathfrak{M}$ which sends every prime ideal into the unique maximal ideal containing it. One of our main results is the fact that μ is continuous; in fact, the pm-rings are also characterized by the presence of a retraction of $\mathcal{P}$ onto $\mathfrak{M}$; this retraction is unique and is precisely the map μ. The preceding results, applied to rings of real valued continuous functions, show that the maximal spectrum of $C(X)$ is a natural model of βX with the consequent possible advantages of introducing βX in this way; e.g. Stone's theorem and the Gelfand-Kolmogoroff theorem follow easily. The methods used also yield results on the mapping $P \to P \cap C^*(X)$ of the prime spectrum of $C(X)$ into the prime spectrum of $C^*(X)$, which extend results obtained in [M] by different techniques.

1. Algebraic results.

1.1. Throughout this paper, the term ring denotes a commutative ring with an identity, all subrings of a given ring contain the identity of the whole ring, and all ring homomorphisms preserve identities. Let A be a ring, and let $J(A)$ be the Jacobson radical (intersection of

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all maximal ideals) of A, $N(A)$ the nilradical (intersection of all prime ideals) of A. Let $\mathcal{P}(A)$ and $\mathfrak{M}(A)$ be the prime and maximal spectra of A, respectively; i.e. $\mathcal{P}(A)$ is the set of all prime ideals of A, topologised by assuming as a base for the closed sets the sets:

$$\mathcal{V}_A(a) = \{P \in \mathcal{P}(A) : a \in P\} \qquad (a \in A)$$

and $\mathfrak{M}(A)$ is the set of all maximal ideals topologised by assuming as a base for the closed sets the sets:

$$V_A(a) = \{M \in \mathfrak{M}(A) : a \in M\} \qquad (a \in A).$$

Of course, $\mathfrak{M}(A)$ is a subspace of $\mathcal{P}(A)$. For $\mathfrak{A} \subseteq \mathcal{P}(A)$,

$$\mathrm{cl}_{\mathcal{P}(A)}(\mathfrak{A}) = \{P \in \mathcal{P}(A) : P \supseteq \bigcap \mathfrak{A}\}.$$

This immediately implies that $\mathfrak{M}(A)$ is T_1 and $\mathcal{P}(A)$ is T_0; in general, neither of these spaces is T_2. Since A has an identity, both $\mathcal{P}(A)$ and $\mathfrak{M}(A)$ are compact.

For each $M \in \mathfrak{M}(A)$, let $\mathcal{P}_M = \{P \in \mathcal{P}(A) : P \subseteq M\}$, $O_M = \bigcap \mathcal{P}_M$. We note in passing that:

$$O_M = \{f \in A : \exists g \notin M \text{ such that } \mathcal{V}_A(f) \supseteq \mathcal{P}(A) - \mathcal{V}_A(g)\}.$$

For, $\mathcal{V}_A(f) \supseteq \mathcal{P}(A) - \mathcal{V}_A(g)$ is equivalent to $fg \in N(A)$, so that the first member contains the second. Next, for $f \in A$, put $S = \{f^n g : n = 0, 1, 2, \cdots, g \in A - M\}$; S is a multiplicative system, hence if $S \cap N = \varnothing$ there is $P \in \mathcal{P}(A)$ contained in $A - S$; then $f \notin P$ and $P \subseteq M$, i.e. $f \notin O_M$.

REMARK. For any $f \in A$, $\mathcal{V}_A(f)$ can be considered as the "zero-set" of f in $\mathcal{P}(A)$. Thus, O_M is the set of all $f \in A$ whose zero-set is a neighborhood of M in $\mathcal{P}(A)$. See [G, 2] and [GJ, 7.12].

A pm-ring is a ring in which every prime ideal is contained in a unique maximal ideal. If A is a pm-ring, denote by μ_A the map $\mathcal{P}(A) \to \mathfrak{M}(A)$ which sends every prime ideal into the unique maximal ideal containing it.

In all these symbols, indication of the ring A is omitted when such a specification is unnecessary.

1.2. THEOREM. *Let A be a ring. The following are equivalent*:

(a) *A is a pm-ring.*

(b) *$\mathfrak{M}$ is a retract of $\mathcal{P}$.*

(c) *For each $M \in \mathfrak{M}$, M is the unique maximal ideal containing O_M (i.e. $\mathcal{P}_M$ is closed in $\mathcal{P}$).*

(d) *$\mathcal{P}$ is a normal space (in general, not T_2).*

Furthermore, if (a) *holds, the map μ $(=\mu_A)$ is the unique retraction of $\mathcal{P}$ onto $\mathfrak{M}$, and $\mathfrak{M}$ is T_2.*

PROOF. The set $\mathcal{P}_M$ is closed in $\mathcal{P}$ if and only if $P \supseteq O_M$ implies $P \subseteq M$. It follows at once that (c) implies (a).

It is also clear that (b) implies (a): let τ be a retraction of $\mathcal{P}$ onto $\mathfrak{M}$, take $P \in \mathcal{P}$ and set $M = \tau P$. Then $P \in \tau^{\leftarrow}[\{M\}]$, and since $\mathfrak{M}$ is T_1, $\tau^{\leftarrow}[\{M\}]$ is closed, so that $\mathrm{cl}_{\mathcal{P}}(\{P\}) \subseteq \tau^{\leftarrow}[\{M\}]$. Therefore, if $M' \in \mathfrak{M}$ and $M' \supseteq P$, then $M' \in \mathrm{cl}_{\mathcal{P}}(\{P\})$, and so $M' = \tau M' = M$, i.e. A is a pm-ring and $\tau = \mu$.

Suppose next that (a) holds. We shall prove that μ is continuous. This will show that (a) implies (b) and (c). Let $\mathfrak{F}$ be a closed subset of $\mathfrak{M}$. Put $F = \bigcap \mathfrak{F}$, $I = \bigcap \{P \in \mathcal{P} : \mu P \in \mathfrak{F}\}$. We must show that $\mu^{\leftarrow}[\mathfrak{F}]$ is closed in $\mathcal{P}$, that is, if $P \in \mathcal{P}$ and $P \supseteq I$, then $\mu P \in \mathfrak{F}$. We first observe that if $Q \in \mathcal{P}$ and $Q \subseteq B = \bigcup \{M : M \in \mathfrak{F}\}$, then $\mu Q \in \mathfrak{F}$. In fact, $Q + F \subseteq B$, hence there exists $M \in \mathfrak{M}$ such that $Q + F \subseteq M$; since $M \supseteq F$, and $\mathfrak{F}$ is closed in $\mathfrak{M}$, $M \in \mathfrak{F}$; and since $M \supseteq Q$, $M = \mu Q$. Now, let $P \in \mathcal{P}$, $P \supseteq I$; we will show that P contains a prime ideal Q contained in B, which implies $\mu P = \mu Q \in \mathfrak{F}$. Put $S = A - B$, $T = A - P$ and pick $s \in S$, $t \in T$; since $P \supseteq I$, there exists $P' \in \mu^{\leftarrow}[\mathfrak{F}]$ such that $t \notin P'$, and since $s \notin P'$, $st \notin P'$, whence $st \notin I$. Thus, the multiplicative system $ST = \{st : s \in S, t \in T\}$ does not meet I, so there exists a prime ideal Q (containing I) disjoint from ST. Of course, $Q \subseteq B$ and $Q \subseteq P$. The continuity of μ is now proved.

Furthermore, if (a) holds, $\mathfrak{M}$ is T_2. For, let M, $M' \in \mathfrak{M}$, $M \neq M'$. The multiplicative system $S = (A - M)(A - M')$ must contain 0; for, otherwise, there would be $P \in \mathcal{P}$ such that $P \cap S = \varnothing$, which implies $P \subseteq M \cap M'$. So there exist disjoint neighborhoods of M and M' (in $\mathcal{P}$, hence also in $\mathfrak{M}$).

Finally, we prove the equivalence of (a) and (d).

(a) implies (d). $\mathfrak{M}$ is compact T_2, μ is continuous and sends closed disjoint subsets of $\mathcal{P}$ into closed disjoint subsets of $\mathfrak{M}$.

(d) implies (a). Let M, $M' \in \mathfrak{M}$, $M \neq M'$. Then $\{M\}$, $\{M'\}$ are closed and disjoint in $\mathcal{P}$. Hence there exist $a \notin M$, $a' \notin M'$ such that $aa' \in N$, whence $M \cap M'$ cannot contain prime ideals.

REMARK. If $J = N$ and $\mathfrak{M}$ is T_2, (a) is true. This can be seen by repeating the last argument in the preceding proof.

1.3. Let $\mathcal{P}^J = \{P \in \mathcal{P} : P \supseteq J\}$; for each $M \in \mathfrak{M}$, let $\mathcal{P}^J_M = \{P \in \mathcal{P} : J \subseteq P \subseteq M\}$ $(= \mathcal{P}_M \cap \mathcal{P}^J)$ and let $O^J_M = \bigcap \mathcal{P}^J_M$.

THEOREM. *Let A be a ring. The following are equivalent*:

(a) *Each prime ideal of A containing J is contained in a unique maximal ideal.*

(b) *$\mathfrak{M}$ is a retract of $\mathcal{P}^J$.*

(c) *For each $M \in \mathfrak{M}$, M is the unique maximal ideal containing O^J_M (i.e. $\mathcal{P}^J_M$ is closed in $\mathcal{P}$).*

(d) $\mathcal{P}^J$ *is a normal space (in general, not* T_2*).*

(e) $\mathfrak{M}$ *is* T_2.

Furthermore, if (a) *holds, the map* $\mu^J:\mathcal{P}^J\to\mathfrak{M}$ *which sends every prime ideal of* $\mathcal{P}^J$ *into the unique maximal ideal containing it, is the unique retraction of* $\mathcal{P}^J$ *onto* $\mathfrak{M}$.

PROOF. Apply Theorem 1.2 to A/J, taking account of the remark in 1.2. (For the equivalence of (c) and (e) see also [GJ].)

REMARK. Note that

$$O_M^J = \{f \in A : \exists g \notin M \text{ such that } V(f) \supseteq \mathfrak{M} - V(g)\}$$

(same argument as in 1.1 for O_M). Hence, by a now obvious terminology, O_M^J is the set of all $f\in A$ whose zero set in $\mathfrak{M}$ is a neighborhood of M in $\mathfrak{M}$.

1.4. We prove in the next paragraph a consequence of the continuity of the map μ. This result will be used in §2. First, we need an elementary lemma.

LEMMA. *Let* A *be a ring,* B *a subring of* A. *For every* $Q\in\mathcal{P}(B)$, *there exists* $P\in\mathcal{P}(A)$ *such that* $P\cap B\subseteq Q$.

PROOF. The set $S=B-Q$ is a multiplicative system of A, and $0\notin S$. Take $P\in\mathcal{P}(A)$ such that $P\cap S=\varnothing$. Then $P\cap B\subseteq Q$.

If R, S are rings and $\eta:R\to S$ is a ring homomorphism, denote by η' the map $\mathcal{P}(S)\to\mathcal{P}(R)$ defined by $\eta'P=\eta^{\leftarrow}[P]$, for every $P\in\mathcal{P}(S)$. Observe that η' is continuous ($\eta'^{\leftarrow}[\mathcal{V}_R(a)]=\mathcal{V}_S(\eta(a))$ for every $a\in R$).

1.5. Assume now that A and B are rings, $\phi:A\to B$ a ring homomorphism, and that $\phi[A]$ is a pm-ring. For every $M\in\mathfrak{M}(B)$ put $\lambda_\phi M=\phi^{\leftarrow}[\mu_{\phi[A]}(M\cap\phi[A])]$. Put also $\mathfrak{M}^\phi(A)=\{M\in\mathfrak{M}(A):M\supseteq\operatorname{Ker}\phi\}$.

THEOREM. *Under the above assumptions,* $\lambda_\phi:\mathfrak{M}(B)\to\mathfrak{M}(A)$ *is a continuous closed map, and its range is* $\mathfrak{M}^\phi(A)$. *Furthermore,* λ_ϕ *is one-to-one if and only if* M, $M'\in\mathfrak{M}(B)$ *and* $M\neq M'$ *imply* $\phi^{\leftarrow}[M]+\phi^{\leftarrow}[M']=A$.

PROOF. Denote by ι the canonical embedding of $\phi[A]$ into B, by η the homomorphism of A onto $\phi[A]$ induced by ϕ. By (1.4), η' and ι' are continuous and since $\lambda_\phi=(\eta'\circ\mu_{\phi[A]}\circ\iota')|\mathfrak{M}(B)$, so is λ_ϕ. It is easily checked that η' maps $\mathfrak{M}(\phi[A])$ homeomorphically onto $\mathfrak{M}^\phi(A)$. By Lemma 1.4 and by the fact that $\phi[A]$ is a pm-ring, $\mu_{\phi[A]}\circ\iota'$ maps $\mathfrak{M}(B)$ onto $\mathfrak{M}(\phi[A])$. Hence $\lambda_\phi[\mathfrak{M}(B)]=\mathfrak{M}^\phi(A)$. Moreover, $\mathfrak{M}^\phi(A)$ is compact T_2, being homeomorphic to $\mathfrak{M}(\phi[A])$, is closed in $\mathfrak{M}(A)$, and $\mathfrak{M}(B)$ is compact. The last statement is obvious.

REMARK. Plainly, a homomorphic image of a pm-ring is a pm-ring. Thus, if A is a pm-ring, the map λ_ϕ can be defined.

1.6. We want to state explicitly a particular case of 1.5 (which, in fact, differs only slightly from the general case).

THEOREM. *Let A be a ring, B a subring of A. Assume that B is a pm-ring. Then the map $\lambda:\mathfrak{M}(A)\to\mathfrak{M}(B)$ which sends every maximal ideal M of A into the unique maximal ideal of B containing $M\cap B$ is a continuous closed map from $\mathfrak{M}(A)$ onto $\mathfrak{M}(B)$; λ is a homeomorphism if and only if M, $M'\in\mathfrak{M}(A)$ and $M\neq M'$ imply $M\cap B+M'\cap B=B$.*

REMARK. Assume $J(A)=N(A)$. By Theorem 1.3 (or Remark 1.2) if λ is a homeomorphism, A is a pm-ring.

2. Applications.

2.1. If T is a topological space, $C(T)$ will be the ring of all continuous real valued functions on T, $C^*(T)$ the subring of all bounded functions of $C(T)$. By [GJ, 14.8], both in $C(T)$ and in $C^*(T)$, the prime ideals containing a given prime ideal form a chain, so that $C(T)$ and $C^*(T)$ are pm-rings. This result does not depend upon first constructing βT. (Caution: in [GJ, 14.8] the result is stated for $C(T)$ only. But clearly, by [GJ, 5.5], the same proof applies as well to $C^*(T)$.) For every $p\in T$, put $M_p(T)=\{f\in C(T):f(p)=0\}$. If T is $T_{3\frac{1}{2}}$ (completely regular and T_1) the map $\theta:p\to M_p(T)$ is a homeomorphism of T onto a dense subspace of $\mathfrak{M}(C(T))$ (for every $f\in C(T)$, $\theta^{\leftarrow}[V_{C(T)}(f)]=\boldsymbol{Z}(f)$ and $\theta[\boldsymbol{Z}(f)]=V_{C(T)}(f)\cap\theta[T]$; by [GJ, 3.2] the sets $\{\boldsymbol{Z}(f):f\in C(T)\}$ are a base for the closed sets of T, and since $C(T)$ separates points, θ is one-to-one). We identify T with $\theta[T]$. If τ is a continuous map on T into a topological space S, τ' denotes the ring homomorphism $C(S)\to C(T)$ given by $\tau'(g)=g\circ\tau$, for every $g\in C(S)$ (see [GJ, 10.2]).

2.2. A theorem of Stone on the extension of continuous mappings is a direct consequence of 1.5.

THEOREM. *Let X and Y be $T_{3\frac{1}{2}}$ spaces, $\tau:X\to Y$ a continuous mapping. The map $\lambda_{\tau'}$* (see 1.5) *is a continuous extension of τ to all of $\mathfrak{M}(C(X))$ into $\mathfrak{M}(C(Y))$.*

PROOF. We only have to show that $\lambda_{\tau'}$ extends τ. This is obvious, since

$$\begin{aligned}\lambda_{\tau'}(M_p(X)) &= \tau'^{\leftarrow}[\mu_{\tau'[C(Y)]}(M_p(X)\cap\tau'[C(Y)])]\\ &= \tau'^{\leftarrow}[M_p(X)\cap\tau'[C(Y)]) = M_{\tau(p)}(Y).\end{aligned}$$

REMARK. This proves that $\mathfrak{M}(C(X))$ is a model of βX (see [GJ, 7N]).

2.3. Let X be a topological space. Put $C=C(X)$, $C^*=C^*(X)$, $\mathfrak{M}=\mathfrak{M}(C)$, $\mathfrak{M}^*=\mathfrak{M}(C^*)$, denote by ι the natural embedding of C^* into C, and denote by σ the map ι' (see 1.4) of $\mathcal{P}=\mathcal{P}(C)$ into $\mathcal{P}^*=\mathcal{P}(C^*)$.

LEMMA. *If M, $M'\in\mathfrak{M}$, $M\neq M'$ then $\sigma M+\sigma M'=C^*$.*

PROOF. Choose $a\in M$, $b\in M'$ in such a way that $a+b=1$. Then $|a|(|a|+|b|)^{-1}\in\sigma M$, $|b|(|a|+|b|)^{-1}\in\sigma M'$ and their sum is 1.

By the preceding lemma, 1.6 implies that $\lambda_\iota=(\mu_{C^*}\circ\sigma)|\mathfrak{M}$ is a homeomorphism of $\mathfrak{M}$ onto $\mathfrak{M}^*$ (cf. [GJ, 7.11]. The same correspondence between $\mathfrak{M}$ and $\mathfrak{M}^*$ is considered in [GHJ, Theorem 6]. The proof makes use of the correspondence between maximal ideals of $C(X)$ and points of βX).

2.4. If A is a pm-ring, Theorem 1.1 implies the following result: Let P_1, $P_2\in\mathcal{P}$. Then $f(P_1)=f(P_2)$ for every $f\in C(\mathcal{P})$ if and only if $\mu P_1=\mu P_2$.

The proof is immediate. By Theorem 1.1, $\mathfrak{M}$ is compact T_2, hence the elements of $C(\mathfrak{M})$ separate the points of $\mathfrak{M}$. Observe also that $\mu P\in\mathrm{cl}_{\mathcal{P}}(\{P\})$ for every $P\in\mathcal{P}$.

The preceding proposition shows that $\mathfrak{M}$ is the $T_{3\frac{1}{2}}$ space obtained from $\mathcal{P}$ by the procedure described in [GJ, 3.9] and that μ is the canonical map constructed in the same reference. In particular, if $A=C(X)$, then $C(\mathcal{P})\cong C(\mathfrak{M})\cong C^*(X)$.

3. Some results concerning prime ideals in $C(X)$ and $C^*(X)$.

3.1. In this section we explore some further properties of the map $\sigma:\mathcal{P}(C(X))\to\mathcal{P}(C^*(X))$ introduced in 2.3, which are of independent interest. Notations are as in 2.3. Denote by U the group of units of C.

LEMMA. *Let $P^*\in\mathcal{P}^*$, $P^*\cap U=\varnothing$. Then $P=P^*U=\{pu:p\in P^*, u\in U\}$ is a prime ideal of C, and $\sigma P=P^*$.*

PROOF. Let $a=pu$, $b=qu$ with p, $q\in P^*$, u, $v\in U$. Then

$$a-b=(pu(1+|u|+|v|)^{-1}-qv(1+|u|+|v|)^{-1})\cdot(1+|u|+|v|)\in P^*U;$$

and if $c\in C$, then

$$ca=(pc(1+|c|)^{-1})u(1+|c|)\in P^*U.$$

Hence P is an ideal of C. If $a\in C^*$, $a(1+|u|)^{-1}=pu(1+|u|)^{-1}\in P^*$; then $a\in P^*$ since $(1+|u|)^{-1}\in C^*-P^*$, and P^* is prime in C^*. Thus

$P \cap C^* \subseteq P^* \subseteq P^* U$, i.e. $P \cap C^* = P^*$. A similar argument shows that P is prime in C.

3.2. We summarise here the results obtained on σ.

THEOREM. (a) *The mapping* $\sigma: \mathcal{P} \to \mathcal{P}^*(\sigma P = P \cap C^*)$ *is a homeomorphism of* $\mathcal{P}$ *onto the subspace of* $\mathcal{P}^*$ *consisting of the prime ideals of* C^* *not containing units of* C.

(b) *For each maximal ideal* M^* *of* C^* *there is a unique maximal ideal* M *of* C *such that* $\sigma M \subseteq M^*$.

Furthermore, if $P^* \in \mathcal{P}^*$, M *and* M^* *are as in* (b), *and* $P^* \subseteq M^*$, *then* P^* *and* σM *are comparable. Specifically*:

(c) *If* P^* *contains a unit of* C, *then* $P^* \supset \sigma M$.

(d) *If* P^* *contains no unit of* C, *then* $P = P^* U$ *is the unique ideal of* C *such that* $P^* = \sigma P$; $P \subseteq M$, *therefore* $P^* \subseteq \sigma M$.

PROOF. (a) Apply Lemma 3.1. For the injectivity, observe that if $\sigma P_1 = \sigma P_2$ and $a \in P_1$ then $a(1+|a|)^{-1} \in \sigma P_1 \subseteq P_2$, whence $a \in P_2$. By symmetry, $P_1 = P_2$.

(b) Uniqueness follows from Lemma 2.3; existence from Theorem 1.6 or by the following argument, which concludes the proof. Use Lemma 1.4 to find $Q \in \mathcal{P}$ such that $\sigma Q \subseteq P^*$, take $M \in \mathfrak{M}$ such that $M \supseteq Q$ and observe that $M^* = \mu_{C^*} P^* = \mu_{C^*}(\sigma Q)$. Since $\sigma M \supseteq \sigma Q$, $\mu_{C^*}(\sigma M) = M^*$, i.e. $\sigma M \subseteq M^*$; and both P^* and σM contain σQ, so that they are comparable.

REMARK 1. Many of these facts are already well known (cf. [GJ_0, Theorem 3.9]). Mandelker proved that if $P^* \in \mathcal{P}^*$, $P^* \subseteq M^* \in \mathfrak{M}^*$, then P^* is comparable to σM, where $M \in \mathfrak{M}$ and $\sigma M \subseteq M^*$, P^* being contained in σM if and only if $P^* \cap U = \varnothing$ [M, 2]. Under additional hypotheses on the underlying space X (X locally compact and σ-compact) he also showed that σ is bijection between prime z-ideals of C and prime z-ideals of C^* disjoint from U [M, Theorem II (a)]. As far as we know, the fact that every prime ideal of C^* disjoint from U is of the form $P \cap C^*$, with $P \in \mathcal{P}$, is new.

REMARK 2. The proofs presented in §3 hold, more generally, if C and C^* are replaced by any rings A and B satisfying the following hypotheses:

(i) A is a lattice ordered ring (commutative and with 1).

(ii) B is an absolutely convex subring of A (and $1 \in B$).

(iii) Every prime ideal of B is convex.

(iv) If $a \in A$, $a \geqq 1$, then a is a unit of A (i.e., A is a convex ring).

(v) For every $a \in A$, $a^2 = |a|^2$ (where $|a| = a \vee (-a)$).

In particular, setting again $A = C(X)$, B can be any absolutely convex subring of A.

References

[G] L. Gillman, *Rings with Hausdorff structure space*, Fund. Math. **45** (1957), 1–16. MR **19**, 1156.

[GHJ] L. Gillman, M. Henriksen and M. Jerison, *On a theorem of Gelfand and Kolmogoroff concerning maximal ideals in rings of continuous functions*, Proc. Amer. Math. Soc. **5** (1954), 447–455. MR **16**, 607.

[GJ_0] L. Gillman and M. Jerison, *Quotient fields of residue class rings of function rings*, Illinois J. Math. **4** (1960), 425–436. MR **23** #A2038.

[GJ] ———, *Rings of continuous functions*, University Series in Higher Math., Van Nostrand, Princeton, N. J., 1960. MR **22** #6994.

[K] J. L. Kelley, *General topology*, Van Nostrand, Princeton, N. J., 1955. MR **16**, 1136.

[M] M. Mandelker, *Prime ideal structure of rings of bounded continuous functions*, Proc. Amer. Math. Soc. **19** (1968), 1432–1438. MR **37** #6761.

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