ON UNIVALENT FUNCTIONS WITH TWO PREASSIGNED VALUES

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Abstract. Let $\mathcal{M}_M$ denote the class of functions analytic and univalent in the unit disc $\Delta$ subject to the conditions

$$f(0) = 0, \quad f(z_0) = z_0, \quad |f'(z)| < M,$$

where $z_0, z_0 \neq 0$, is a fixed point of $\Delta$ and $1 \leq M < \infty$.

In the present note, we determine by the method of circular symmetrization, the exact value of the "Koebe constant" for the class $\mathcal{M}_M$. We also determine Koebe sets for the class $\mathcal{M}_M^*$ consisting of the starlike functions, and for $\mathcal{M}_M^*$, consisting of all functions mapping $\Delta$ onto domains convex in the direction $e^{ia}$.

By "Koebe set" we understand the set $K(\mathcal{M}_M), K(\mathcal{M}_M^*) = \bigcap_{f \in \mathcal{M}_M} f(\Delta)$.

1. Introduction. Let $\mathcal{M}_M$ denote the class of functions analytic and univalent in the unit disc $\Delta$ subject to the conditions

$$(1) \quad f(0) = 0, \quad f(z_0) = z_0, \quad |f'(z)| < M$$

where $z_0, z_0 \neq 0$, is a fixed point of $\Delta$ and $1 < M < \infty$.

So far as we know there are no results concerning class $\mathcal{M}_M$, whereas many authors have considered certain extremal problems in the case $M = \infty$.

In the present note, we determine by the method of the circular symmetrization, the exact value of the Koebe constant for the class $\mathcal{M}_M$. We also determine Koebe sets for the class $\mathcal{M}_M^*$, consisting of the starlike functions, and for $\mathcal{M}_M^*$, consisting of all functions mapping $\Delta$ onto domains convex in the direction $e^{ia}$.

By "Koebe set" [1] we understand the set $K(\mathcal{M}_M), K(\mathcal{M}_M^*) = \bigcap_{f \in \mathcal{M}_M} f(\Delta)$.

2. Main results. 2.1. We start with the determination of the Koebe constant for the class $\mathcal{M}_M$. 

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Presented to the Society, November 28, 1970; received by the editors July 27, 1970.

AMS 1969 subject classifications. Primary 3042, 3052.

Key words and phrases. Univalent function, Koebe constant, Koebe domain, starlike mapping, convex in one direction, hyperbolic measure, Green's function, bounded functions.

1 The first author acknowledges support received under National Science Foundation Grant GP-11158.

2 The second author acknowledges support received from I.R.E.X.
Theorem 1. The image-domain of $\Delta$ under each function of the class $\mathfrak{M}_M$ contains the disc $|w| < r(M)$ where

\[ r(M) = 2\delta^2 - M - 2\delta(\delta^2 - M)^{1/2}, \]
\[ \delta = (M - |z_0|)(1 - |z_0|)^{-1}. \]

The number $r(M)$ cannot be replaced by any greater number without additional assumptions on the functions of the class $\mathfrak{M}_M$.

Proof. Suppose that $\Omega = f(\Delta)$ and $\rho(a, b, \Omega)$ denotes the hyperbolic distance of the points $a$, $b$ with respect to the domain $\Omega$.

Since the hyperbolic distance is a conformal invariant, we have

\[ \text{arc th } |z_0| = \rho(0, z_0, f(\Delta)), \quad f \in \mathfrak{M}_M. \]

Let $\Omega^*$ be a domain obtained from $\Omega$ by using circular symmetrization with respect to the ray $[0, z_0, \infty)$. It is clear that the origin and $z_0$ are in $\Omega^*$. Moreover, it is well known that circular symmetrization decreases the hyperbolic distance, so that

\[ \rho(0, z_0, \Omega^*) \leq \rho(0, z_0, \Omega) \quad (3) \]

holds. However, each domain $\Omega^*$ is contained in a domain $D_h$, $D_h = \Delta_M \setminus [-M e^{i\alpha}, -he^{i\alpha}]$ for some $h > 0$, where $\Delta_M$ is the disc $|w| < M$. Hence

\[ \rho(0, z_0, \Omega^*) \leq \rho(0, z_0, D_h) \quad (4) \]

The conditions (2), (3) and (4) now yield our basic inequality

\[ \text{arc th } |z_0| \leq \rho(0, z_0, D_h) \quad (5) \]

For our purpose it is sufficient to consider $z_0 = r$. Now let $\varphi(z)$, $\varphi(0) = 0$ be the function that maps the domain $D_h$, which is now $\Delta_M \setminus [-M, -h]$ conformally onto the unit disc. Then we have

\[ \rho(0, z_0, D_h) = \text{arc th } |\varphi(r)|, \]

where

\[ \varphi(r) = \frac{2Mr}{d(r^2 + M^2) + 2Mr(1 - d) + ([d(M - r)^2 + 2Mr] - 4M^2)^{1/2}}, \]
\[ d = 4hM(M + h)^{-2}. \]

Thus (5) yields the inequality

\[ d(M - r)^2 + 2Mr + ([d(M - r)^2 + 2Mr] - 4M^2)^{1/2} \geq 2M, \]

which holds if
or
\[ h \geq 2\delta^2 - M - 2\delta(\delta^2 - M)^{1/2}, \quad \delta = (M - r)(1 - r)^{-1}. \]

The sign of equality in these last considerations occurs for the function \( f(z) \) given by the equation
\[
\frac{f(z)}{[M - f(z)]^2} = \left(\frac{1 - r}{M - r}\right)^2 \frac{z}{(1 - z)^2}.
\]

This completes our proof.

**Corollary 1.** For the cases \( M = \infty \) and \( z_0 = 0 \) we have
\[
r(\infty) = \frac{1}{4}(1 - |z_0|^2), \quad r_0(M) = 2M^2 - M - 2M(M^2 - M)^{1/2},
\]
respectively. Both constants are well known ([3], [4], respectively).

2.2. Now we shall be concerned with the determination of Koebe sets for starlike functions of the class \( \mathfrak{M}_M \). Let \( \Delta_M \) be the disc \( \{w: |w| < M\} \) and let \( G \) be the family of all its subdomains that are (i) starshaped with respect to the origin and (ii) contain the fixed point \( z_0 \). For \( D \in G \), let \( g(w, z_0, D) \) be the Green’s function with pole at \( z_0 \), and let \( \mu(w_0) \) be defined as a solution of the following extremal problem:
\[
\mu(w_0) = \sup_{D \in G_{z_0} \Delta_M} g(0, z_0, D).
\]

The following formula for \( \mathcal{K}(\mathfrak{M}_M) \) was obtained by Krzyż and Zlotkiewicz [2]:
\[
\mathcal{K}(\mathfrak{M}_M) = \{w: \mu(w) < - \log |z_0|\}.
\]

Hence, in order to determine \( \mathcal{K}(\mathfrak{M}_M) \) it is sufficient to solve the extremal problem (6). We shall use (7) for the two classes \( \mathfrak{M}_M^*, \mathfrak{M}_M^\star \) consisting of all starlike functions of the class \( \mathfrak{M}_M \) and of those functions mapping \( \Delta \) onto domains convex in the given direction \( e^{i\alpha} \), respectively.

**Theorem 2.** The Koebe set of the class \( \mathfrak{M}_M^* \) is determined by the condition
\[
|w - z_0| \frac{|M - wz_0|}{(M + |w|)^2} + \left(\frac{Mw + z_0}{M + |w_0|}\right)^2 \frac{1}{|w|} < \frac{1}{2}(1 + |z_0|^2).
\]

The limit case, \( M = \infty \), is
\[ |w - z_0| + |w| < \frac{1}{2} (1 + |z_0|^2). \]

**Proof.** We shall solve the extremal problem (6). Let \( w_0 \in \Delta_M \) and let \( D \in G \) such that \( w_0 \in \Delta_M \setminus D \). Then there exists a domain \( D_{w_0} = \Delta_M \setminus (Me^{i\varphi}, w_0) \) such that \( g(0, z_0, D_{w_0}) \geq g(0, z_0, D) \). In order to determine the function \( \mu(w) \) we map \( D_{w_0} \) onto the upper half-plane \( U \). The corresponding transformation is

\[
W = (z^2 + h^2)^{1/2}, \quad \zeta = i(Me^{i\varphi} - w)(Me^{i\varphi} + w)^{-1}
\]

where

\[
W(0) = i(1 - h^2)^{1/2}, \quad W(z_0) = i(\zeta(z_0) - h^2)^{1/2},
\]

\[
h = (M - |w_0|)(M + |w_0|)^{-1}.
\]

Using the conformal invariance of Green's function, we obtain

\[
g(0, z_0, D_{w_0}) = g(W(0), W(z_0), U) = \log \left| \frac{W(0) - W(z_0)}{W(0) - W(z_0)} \right|
\]

which with (7) gives us

\[
\frac{|h^2 - W^2(z_0)| + 1 - h^2}{|1 - W^2(z_0)|} < \frac{1}{2} \left( |z_0| + |z_0|^{-1} \right).
\]

This last inequality reduces to (8) after some simple substitutions.

If we let \( M \to \infty \) in (8), then we obtain the well-known elliptic domain defined by (9) [5]. This completes the proof of Theorem 2.

**Remark.** We may apply the method given by (6) and (7) to determine Koebe sets for some other subclasses of \( \mathbb{M}_M \). One can convince himself that the extremal domain for the class of convex maps consists of domains whose boundary is an arc of \( \partial \Delta_M \) plus a chord of \( \Delta_M \). For the close-to-convex maps the extremal domains are those obtained from \( \Delta_M \) by removing a slit along a segment emanating from \( w_0 \). Unfortunately, the corresponding formulas for the Green's functions involve transcendental functions that have made it impossible for us to find a suitable description of the Koebe sets for those classes.

However, we can establish the following result.

**Theorem 3.** The set \( \mathcal{K}(\mathbb{M}_M^2) \) is given by the condition

\[
1 + [A (1 + \cos 2\theta) + (B^2 - 1) \cos 2\theta + BC \sin 2\theta]^{1/2} < [1 - (1 - D^2)^{1/2}] |z_0|^{-2}
\]

where
A = 2hd(h + d)^{-2}, \quad B = |h - d| |z_0|^{-1},
D = \frac{|z_0|}{|h + d|^{-1}}, \quad C = [(1 - D^2)(D^2 - 1 + 2A)]^{1/2},
\quad h = |w|, \quad d = |w - z_0|, \quad \theta = \alpha - \arg z_0.

The set $K(z_0)$ is a simply connected Jordan domain if $|z_0| < (1 + |\sin \theta|)^{-1/2}$, $\theta \neq 0, \pi$ and is a union of two simply connected Jordan domains which are symmetric with respect to the point $z_0/2$ if $(1 + |\sin \theta|)^{-1/2} < |z_0|$, $\theta \neq 0, \pi$.

**Proof.** Let $D$ be a domain that is convex in the direction $e^{ia}$ containing the origin and $z_0$, and omitting a given point $w_0$, and let $E^2$ denote the open plane. There exists a ray $l_a$ such that $D \subseteq E^2 \setminus l_a$ and $g(0, z_0, D) \leq g(0, z_0, E^2 \setminus l_a)$. Hence we can restrict ourselves to the domains $D_a = E^2 \setminus l_a$ without loss in generality. Now by a translation and a rotation we can send $l_a$ to the negative real axis and the origin and $z_0$ to the points $de^{i(\varphi - a)}$ and $he^{i(\psi - a)}$, respectively. The preceding transformation followed by the transformation $w^{1/2}$ gives us the right half-plane $H$. Hence we have

$$g(0, z_0, D) = g(d^{1/2}e^{i(\varphi - a)/2}, h^{1/2}e^{i(\psi - a)/2}, H)$$

$$= \frac{1}{2} \log \frac{d + h + 2(hd)^{1/2} \cos \frac{\alpha - \varphi + \psi}{2}}{d + h - 2(dh)^{1/2} \cos \frac{1}{2}(\varphi - \psi)} = F(\varphi, \psi)$$

where $h$, $d$, $\alpha$ are fixed and $\varphi$, $\psi$ have to be chosen so that $F(\varphi, \psi)$ is a maximum. It is geometrically clear that

$$2hd \cos (\varphi - \psi) = d^2 + h^2 - |z_0|^2$$

and

$$h \cos \varphi - d \cos \psi = |z_0| \cos \theta$$

hold. From (12) and (13), after some elementary calculations, we obtain

$$4hd \cos^2 \left( \frac{\alpha - \varphi + \psi}{2} \right) = \frac{(h^2 - d^2) \cos 2\theta - |z_0|^2(h^2 + d^2) \cos 2\theta}{|z_0|^2} + 2hd \left| z_0 \right|^2 - |h^2 - d^2| \sin 2\theta$$

Now (11), (12) and (14) give us the formula (10).
It is clear that (10) is symmetric with respect to the $h$ and $d$. This means that (10) is symmetric with respect to the point $z_0/2$. However, $z_0/2 \in \mathcal{K}(\mathbb{R}^+)$ if and only if $|z_0|^2 < (1 + |\sin \theta|)^{-1} = K$. Of course, $K < 1$ if $\theta \neq 0, \pi$. Hence, if $|z_0|^2 < K$ then $\mathcal{K}(\mathbb{R}^+)$ is a simply connected Jordan domain; it is the union of two simply connected domains if $|z_0|^2 > K$. Our proof is now complete.

If $z_0 = 0$ and $\theta = \pi/2$ we obtain

$$8 |w| (|w| + |\text{Im} w|) < 1,$$

which defines the Koebe set for the class of univalent functions mapping $\Delta$ onto domains convex in the direction of the imaginary axis and normalized by the conditions $f(0) = 0$, $f'(0) = 1$.

References


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