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EXTENDING UNIFORMLY CONTINUOUS PSEUDO-ULTRAMETRICS AND UNIFORM RETRACTS

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ABSTRACT. It is first proved that any uniformly continuous pseudo-ultrametric on a subspace of a non-Archimedean uniform space X has a uniformly continuous extension to X (which preserves total boundedness or separability). Then it is proved that every complete subspace of an ultrametrizable space X is a uniform retract of X. This has consequences concerning the extension of uniformly continuous functions.

Isbell [6, Lemma 1.4] proved that every bounded uniformly continuous pseudometric ρ on a subspace of a uniform space X has an extension to a bounded uniformly continuous pseudometric on X. In this paper we consider the same problem when X is a non-Archimedean uniform space (see [7] for terminology) and ρ is a pseudo-ultrametric, i.e., ρ satisfies the strong triangle inequality $\rho(x, y) \leqq \max\{\rho(x, z), \rho(z, y)\}$. In this connection we mention that the topology of any Hausdorff space of small inductive dimension zero is defined by a non-Archimedean uniform structure. It is proved that, even if ρ is not bounded, it has an extension to a uniformly continuous pseudo-ultrametric on X. There is also interest [1], [2], [5] in extending pseudometrics which are either totally bounded or separable, and we prove that if ρ has one of these properties, then it has a uniformly continuous extension which has the same property.

The above result is obtained as a simple corollary of a theorem which states that if ρ is a uniformly continuous pseudo-ultrametric on a subset S of a non-Archimedean uniform space X, then there is a uniformly continuous extension $\Phi: X \to S^*$ of the canonical map $\phi: S \to S^*$ of S into its completion S^* for ρ. We say that a subset S of a uniform space X is a *uniform retract* of X if there is a uniformly continuous function from X into S which extends the identity function on S. Another corollary is that any complete subset of a pseudo-ultrametric space X is a uniform retract of X. This has implications concerning the extension of uniformly continuous functions.

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LEMMA. *Let S be a subset of a non-Archimedean uniform space X. If V is a non-Archimedean entourage on S, there is a non-Archimedean entourage V' on X such that $V'\cap(S\times S)=V$.*

PROOF. Let W be an entourage on X such that $W\cap(S\times S)=V$ and let T be a non-Archimedean entourage on X contained in W. Let

$$V' = \bigcup_{n=1}^{\infty} (T \cup V)^n.$$

Then V' is clearly a non-Archimedean entourage on X. We will prove that $V'\cap(S\times S)=V$ by showing by induction on n that any set of the form $A_1 \circ A_2 \circ \cdots \circ A_n$, where each A_i is either T or V, intersects $S\times S$ inside V. This is clear if either $n=1$ or each $A_i=T$. If $n>1$, $A_{i_0}=V$, and $(s_1, s_{n+1})\in A_1\circ A_2 \circ \cdots \circ A_n \cap (S\times S)$, then $(s_1, s_{n+1})=(s_1, x_2)\circ(x_2, x_3)\circ\cdots\circ(x_n, s_{n+1})$, where $(x_i, x_{i+1})\in A_i$ $(x_1=s_1,\ x_{n+1}=s_{n+1})$. Since $A_{i_0}=V$, x_{i_0} and $x_{i_0+1}\in S$. By the inductive hypothesis, if $i_0\neq n$, then $(s_1, x_2)\circ(x_2, x_3)\circ\cdots\circ(x_{i_0}, x_{i_0+1})$ and $(x_{i_0+1}, x_{i_0+2})\circ\cdots\circ(x_n, s_{n+1})$ belong to V and so $(s_1, s_{n+1})\in V^2=V$. Similarly, if $i_0=n$, then $(s_1, x_2)\circ(x_2, x_3)\circ\cdots\circ(x_{n-1}, x_n)$ and $(x_n, s_{n+1})\in V$ and so $(s_1, s_{n+1})\in V^2=V$.

THEOREM 1. *Let ρ be a uniformly continuous pseudo-ultrametric on a subset S of a non-Archimedean uniform space X. Then there is a uniformly continuous function $\Phi: X\to S^*$ which extends the canonical map $\phi: S\to S^*$, where (S^*, ρ^*) is the completion of (S, ρ).*

PROOF. For each integer $n\geqq 1$, $\{(x, y)\mid \rho(x, y)<1/n\}$ is a symmetric non-Archimedean entourage on S. By the lemma there is a sequence $\{V_n \mid n\geqq 1\}$ of symmetric non-Archimedean entourages on X such that $V_1=X\times X$, $V_{n+1}\subseteq V_n$, and for $n>1$, $V_n\cap(S\times S)=\{(x, y) \mid p(x, y)<1/n\}$. Then $\mathcal{V}_n=\{V_n(s)\mid s\in S\}$ is a partition of $V_n(S)$ [4, Lemma 5.3]. Let $\{s_i\mid i\in I_n\}\subseteq S$ be a complete set of representatives of $\mathcal{V}_n$. Define a function N from X into $\boldsymbol{N}\cup\{\infty\}$ by

$$N(x) = \sup\{n \mid x \in V_n(S)\}.$$

If n is an integer and $1\leqq n\leqq N(x)$ let $i(n, x)\in I_n$ be such that $V_n(x)=V_n(s_{i(n,x)})$. For each integer $n\geqq 1$ define a function $f_n: X\to S$ by

$$\begin{aligned} f_n(x) &= s_{i(N(x),x)}, \qquad \text{if } N(x) < \infty, \\ &= s_{i(n,x)}, \qquad \text{if } N(x) = \infty. \end{aligned}$$

We first show that for all $x\in X$, $\{f_n(x)\mid n\geqq 1\}$ is a Cauchy sequence, which is trivial if $N(x)<\infty$. If $N(x)=\infty$ and $m\geqq n>1$, then

$(f_m(x), f_n(x)) = (s_{i(m,x)}, x) \circ (x, s_{i(n,x)}) \in V_m \circ V_n = V_n$ and so $\rho(f_m(x), f_n(x)) < 1/n$. Define $\Phi: X \to S^*$ by letting $\Phi(x)$ be the equivalence class in S^* of the Cauchy sequence $\{f_n(x) \mid n \geqq 1\}$. Then Φ extends ϕ because if $s \in S$, then $N(s) = \infty$ and for each $n \geqq 2$, $(s, f_n(s)) \in V_n$ and so $\rho(s, f_n(s)) < 1/n$.

We now show that Φ is uniformly continuous by showing that for $n \geqq 2$ and $(x, y) \in V_n$ we have $\rho^*(\Phi(x), \Phi(y)) \leqq 1/n$.

Case (i). $N(x) < \infty$, $N(y) < \infty$. Then it is impossible that n is strictly between $N(x)$ and $N(y)$, for if $N(x) < n < N(y)$, then $y \in V_n(S)$ and $(x, y) \in V_n$, from which it follows that $x \in V_n(S)$, a contradiction to $N(x) < n$. If n is not smaller than $N(x)$ and $N(y)$, then $(x, y) \in V_n \subseteq V_{N(x)} \cap V_{N(y)}$ and so $N(x) = N(y)$, and $\Phi(x) = \Phi(y)$. If n is not larger than $N(x)$ and $N(y)$, then, for each $r \geqq 1$, $(f_r(x), f_r(y)) = (s_{i(N(x),x)}, x) \circ (x, y) \circ (y, s_{i(N(y),y)}) \in V_{N(x)} \circ V_n \circ V_{N(x)} \subseteq V_n$ and so $\rho(f_r(x), f_r(y)) < 1/n$. From this it follows that $\rho^*(\Phi(x), \Phi(y)) \leqq 1/n$.

Case (ii). $N(x) < \infty$, $N(y) = \infty$. Then $x \in V_n(S)$ since $(x, y) \in V_n$ and $y \in V_n(S)$. Therefore, $N(x) \geqq n$. Then for $m \geqq n$, $(y, f_m(y)) \in V_m \subseteq V_n$, $(x, y) \in V_n$, and $(x, f_m(x)) \in V_{N(x)} \subseteq V_n$ and so $(f_m(x), f_m(y)) \in Vn$. Consequently, $\rho(f_m(x), f_m(y)) < 1/n$ and so $\rho^*(\Phi(x), \Phi(y)) \leqq 1/n$.

Case (iii). $N(x) = N(y) = \infty$: For $m \geqq n$, $(x, f_m(x)) \in V_m$, $(x, y) \in V_n$, $(y, f_m(y)) \in V_m$, and so $(f_m(x), f_m(y)) \in V_n$. As before, $\rho^*(\Phi(x), \Phi(y)) \leqq 1/n$.

THEOREM 2. *Let S be a subset of a non-Archimedean uniform space X and let ρ be a uniformly continuous pseudo-ultrametric on S. Then there is a uniformly continuous pseudo-ultrametric d on X which extends ρ.*

Moreover, if S is either totally bounded or separable for ρ, then d can be chosen so that X has the same property for d.

PROOF. Using the notation of Theorem 1, define d on $X \times X$ by $d(x, y) = \rho^*(\Phi(x), \Phi(y))$. Since Φ is uniformly continuous, so is d.

If S is totally bounded for ρ, then for each $n \geqq 2$, I_n is finite and $\{s_i \mid i \in I_n\}$ is $(1/n)$-dense in S for ρ. From the definition of Φ it is easy to see that $\{s_i \mid i \in I_n\}$ is also $(1/n)$-dense in X for d. If S is separable for ρ, then each I_n is countable and $\bigcup_{n=1}^{\infty} \{s_i \mid i \in I_n\}$ is dense in S for ρ and therefore also in X for d.

THEOREM 3. *Let S be a subspace of a non-Archimedean uniform space X and let ρ be a uniformly continuous pseudo-ultrametric on S making S complete. Then there is a function $f: X \to S$ which is uniformly continuous (for the given uniform structure on X and the uniform structure on S defined by ρ) and extends the identity function on S.*

PROOF. In this case $\phi: S \to S^*$ is surjective. Define $f: X \to S$ by choosing for each $x \in X$ an element $f(x) \in \phi^{-1}(\Phi(x))$ with $f(s) = s$ if $s \in S$. Then for all $x, y \in X$,

$$\rho(f(x), f(y)) = \rho^*(\Phi(x), \Phi(y)) = d(x, y).$$

Since d is uniformly continuous, so is f.

COROLLARY 1. *Every complete subset of a pseudo-ultrametric space X is a uniform retract of X.*

COROLLARY 2. *Every uniformly continuous function from a complete subspace of a pseudo-ultrametric space X into a uniform space has a uniformly continuous extension to X.*

COROLLARY 3. *Every uniformly continuous function from a subspace of a pseudo-ultrametric space X into a complete Hausdorff uniform space has a uniformly continuous extension to X.*

PROOF. This follows immediately from Corollary 2 and [**3**, Theorem 2, p. 190] by first extending to the completion of S, then extending to the completion of X, and finally restricting to X.[2]

REFERENCES

1. R. A. Alo and H. L. Shapiro, *Extensions of totally bounded pseudometrics*, Proc. Amer. Math. Soc. **19** (1968), 877–884. MR **38** #667.

2. R. F. Arens, *Extension of coverings, of pseudometrics, and of linear-space-valued mappings*, Canad. J. Math. **5** (1953), 211–215. MR **14**, 1108.

3. N. Bourbaki, *Elements of mathematics. General topology*. Part I, Hermann, Paris; Addison-Wesley, Reading, Mass., 1966. MR **34** #5044a.

4. R. Ellis, *Extending continuous functions on zero-dimensional spaces*, Math. Ann. **186** (1970), 114–122.

5. T. E. Gantner, *Extensions of uniformly continuous pseudometrics*, Trans. Amer. Math. Soc. **132** (1968), 147–157. MR **36** #5886.

6. J. R. Isbell, *On finite-dimensional uniform spaces*, Pacific J. Math. **9** (1959), 107–121. MR **21** #4407.

7. A. F. Monna, *Remarques sur les mêtriques non-archimêdiennes*. I, Nederl. Akad. Wetensch. Proc. **53**, 470–481 = Indag. Math. **12** (1950), 122–133. MR **12**, 41.

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