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EFFECTIVELY MINIMIZING EFFECTIVE FIXED-POINTS

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ABSTRACT. This note answers an open problem posed by H. Rogers, Jr. on p. 202 of *Theory of recursive functions and effective computability* by proving the following invariant form of one of his results [op. cit., p. 200, Theorem XIV]: for any fixed-point function $\hat{n}$ there exists a recursive function g such that if z is an index of an effective operator Ψ, then $g(z)$ is also an index of Ψ, and $\hat{n}(g(z))$ is an index of the minimum fixed-point of Ψ with respect to inclusion.

Let W and φ denote standard enumerations of the recursively enumerable sets and the partial recursive functions, respectively. Let $\hat{n}$ be a recursive fixed-point function for W (i.e. if $\hat{n}(z)$ is in the domain of φ_z then $\hat{n}(z)$ and $\varphi_z\hat{n}(z)$ are W-indices of the same set, while otherwise $\hat{n}(z)$ is a W-index of the empty set). Clearly, if z is an index of an effective operator Ψ (i.e. for all x, $\varphi_z(x)$ is a W-index of $\Psi(W_x)$), then $\hat{n}(z)$ is a W-index of a fixed-point of Ψ; unfortunately, it is not always a W-index of the minimum fixed point of Ψ (cf. [1, p. 196]). However, *there exists a recursive function g such that if z is an index of an effective operator Ψ, then*

(i) *$g(z)$ is also an index of Ψ, and*

(ii) *$\hat{n}g(z)$ is a W-index of the minimum fixed-point of Ψ.*

PROOF. Let f be a recursive function whose range is $\{\langle x, y\rangle \mid y \in W_x\}$. Abusing language, we say that y is in W_x before y' is in $W_{x'}$ if there exists a number n such that $\langle x, y\rangle \in \{f(1), \cdots, f(n)\}$ but $\langle x', y'\rangle \notin \{f(1), \cdots, f(n)\}$.

We use the Recursion Theorem to obtain recursive functions k, h, and g such that

$$W_{k(z,x)} = \{y \mid y \text{ is in } W_{h(z)} \text{ before } x \text{ is in } W_{\varphi_z k(z,x)}\},$$
$$W_{h(z)} = \{x \mid x \text{ is in } W_{\varphi_z k(z,x)} \text{ before } x \text{ is in } W_{h(z)}\},$$
$$\varphi_{g(z)} = \lambda x[\varphi_z(x) \text{ if } x \neq \hat{n}g(z);\ h(z) \text{ if } x = \hat{n}g(z)].$$

Now suppose that z is an index of an operator Ψ. Then $\Psi(W_{h(z)}) \subset W_{h(z)}$, since if $x \in \Psi(W_{h(z)})$, then $x \in \Psi(W_{k(z,x)}) = W_{\varphi_z k(z,x)}$ (for otherwise $W_{k(z,x)} = W_{h(z)}$ while $x \in \Psi(W_{h(z)})$) and hence $x \in W_{h(z)}$

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(for otherwise we violate the definition of $W_{h(z)}$). Conversely, $W_{h(z)} \subset \Psi(W_{h(z)})$ for if $x \in W_{h(z)}$, then $x \in W_{\varphi_z k(z,x)} = \Psi(W_{k(z,x)})$ and hence $x \in \Psi(W_{h(z)})$ (since $W_{k(z,x)} \subset W_{h(z)}$ and Ψ is monotonic by the Myhill-Shepherdson Theorem [**1**, p. 395]). Thus $W_{h(z)}$ is a fixed-point of Ψ.

To see that $W_{h(z)}$ is the minimum fixed-point of Ψ, suppose that A is a fixed-point of Ψ, that $x \in W_{h(z)}$, and that all numbers which are in $W_{h(z)}$ before x are members of A. Then $W_{k(z,x)} \subset A$, for if $y \in W_{k(z,x)}$, then y is in $W_{h(z)}$ before x is in $W_{\varphi_z k(z,x)}$ which is before x is in $W_{h(z)}$. Thus $\Psi(W_{k(z,x)}) \subset \Psi(A) = A$. But, as already noted, $x \in \Psi(W_{k(z,x)})$ whenever $x \in W_{h(z)}$. Thus $x \in A$. So, by induction, $W_{h(z)} \subset A$.

Thus (ii) is satisfied since by our choice of g,

$$W_{\hat{n}g(z)} = W_{\varphi_{g(z)}(\hat{n}g(z))} = W_{h(z)}.$$

We see that (i) is also satisfied, since in the case where $x = \hat{n}g(z)$,

$$\begin{aligned} W_{\varphi_{g(z)}(x)} &= W_{\varphi_{g(z)}(\hat{n}g(z))} = W_{\hat{n}g(z)} = \Psi(W_{\hat{n}g(z)}) \\ &= W_{\varphi_z(\hat{n}g(z))} = W_{\varphi_z(x)} = \Psi(W_x), \end{aligned}$$

while in the case where $x \neq \hat{n}g(z)$, $\varphi_{g(z)}(x) = \varphi_z(x)$ and hence $W_{\varphi_{g(z)}(x)} = W_{\varphi_z(x)} = \Psi(W_x)$. $\square$

It should be noted that if one identifies partial recursive functions with their graphs, then the above discussion remains valid if "W" is replaced by "φ".

References

1. Hartley Rogers, Jr., *Theory of recursive functions and effective computability*, McGraw-Hill, New York, 1967. MR **37** #61.

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