OSCILLATION THEOREMS FOR SECOND-ORDER
DIFFERENTIAL EQUATIONS WITH FUNCTIONAL ARGUMENTS

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Abstract. The oscillatory behavior of \( Y''(t) + P(t) Y(g(t)) = 0 \)
where \( g(t) \to \infty \) as \( t \to \infty \) is investigated. Sufficient conditions
for the oscillation of \( Y'(t) \) and \( Y(t) \) are developed.

1. Introduction. The purpose of this paper is to study the oscillatory
behavior of the differential equation

\[
Y''(t) + P(t) Y(g(t)) = 0.
\]

It is assumed throughout that \( P(t) \) and \( g(t) \) are continuous on \([a, \infty)\) and
that \( g(t) \to \infty \) as \( t \to \infty \). A nontrivial solution to (1.1) is called oscillatory
if it exists on a half-line and has arbitrarily large zeros. A solution is
called nonoscillatory if it is eventually of one sign. Equation (1.1) is
called oscillatory if every solution is oscillatory.

2. Main results. Waltman [5] and Bradley [2] have recently demon-
strated that all solutions of (1.1) are oscillatory if \( P(t) \geq 0 \) and \( \int_0^\infty P(t) \, dt = \infty \).

If \( g(t) \equiv t \), equation (1.1) reduces to the ordinary differential equation

\[
Y''(t) + P(t) Y(t) = 0.
\]

It is a well-known theorem of Wintner [6] and Leighton [3] that \( \int_0^\infty P(t) \, dt = \infty \) is sufficient for equation (2.1) to be oscillatory even when
\( P(t) \) is not assumed positive. It is also known [1] that all solutions of the
nonlinear equation

\[
Y''(t) + P(t) h(Y(t)) = 0,
\]

\( h(x)x > 0 \) for \( x \neq 0 \),

\( h'(x) \geq 0 \) for all \( x \),

oscillates if \( \int_0^\infty P(t) = \infty \), where \( P(t) \) is not assumed positive. On the basis
of these two results, one might conjecture that the Leighton-Wintner oscillation theorem could be extended to equations of the form (1.1). The equation $Y''(t) + (\sin t)/(2 - \sin t))Y(t - \pi) = 0$ demonstrates that this is not the case. This equation has the nonoscillatory solution $Y(t) = 2 + \sin t$ even though $\int_{-\infty}^{\infty} ((\sin t)/(2 - \sin t)) \, dt = \infty$. However this solution does have the property that $Y'(t)$ has arbitrarily large zeros. Theorem 2.1 below shows that the derivative of any solution of an equation of form (1.1) has arbitrarily large zeros if we assume $g'(t) \geq 0$ and $\int_{-\infty}^{\infty} P(t) \, dt = \infty$.

**Theorem 2.1.** Assume $g(t)$ is differentiable, $g'(t) \geq 0$, and $\int_{-\infty}^{\infty} P(t) \, dt = \infty$. Then $Y'(t)$ has arbitrarily large zeros for any solution $Y(t)$ of (1.1).

**Proof.** If $Y(t)$ oscillates then the conclusion of the theorem is certainly true. If $Y(t)$ is ultimately positive, then so is $Y(g(t))$. Suppose $Y'(t) > 0$ for all large $t$. Then $W(t) = Y'(t)/Y(g(t))$ satisfies the equation

$$W'(t) = -P(t) - W(t) \frac{Y'(g(t))}{Y(g(t))} g'(t) \leq -P(t).$$

Integrating we obtain

$$W(x) \leq W(x) - \int_{x}^{\infty} P(t) \, dt.$$

Since $\int_{-\infty}^{\infty} P(t) \, dt = \infty$ we have a contradiction. Suppose now that $Y'(t) < 0$ for all large $t$. It is not difficult to see that $\int_{-\infty}^{\infty} P(t) \, dt = \infty$ implies that there exists $T$ sufficiently large so that

$$\int_{T}^{t} P(s) \, ds \geq 0 \quad \text{for} \ t \geq T.$$

Hence, we have

$$\int_{T}^{t} P(s)Y(g(s)) \, ds = Y(g(t)) \int_{T}^{t} P(s) \, ds - \int_{T}^{t} Y'(g(s))g'(s) \int_{T}^{s} P(r) \, dr \, ds \geq 0, \quad t \geq T.$$

Now integrating equation (1.1) we have $Y'(t) \leq Y'(T) < 0$ which contradicts the fact that $Y(t)$ is positive for large $t$. This completes the proof.

Considering the equation

$$(2.2) \quad Y''(t) + \lambda P(t) Y(t) = 0,$$

we shall call $P(t)$ a strongly oscillatory coefficient if (2.2) is oscillatory for all positive $\lambda$. If $P(t) \geq 0$, Nehari [4] shows that

$$\lim_{x \to \infty} \, \int_{x}^{\infty} P(t) \, dt = \infty.$$
is a necessary and sufficient condition for \( P(t) \) to be a strongly oscillatory coefficient.

In Theorem 2.2 below, we shall consider the equation

\[
Y''(t) + P(t)f(Y(t), Y(g(t))) = 0,
\]

where \( P, g, \) and \( f \) are assumed continuous, and \( f(y, W) \) has the sign of \( y \) and \( W \) when they have the same sign. We will show that if \( P(t) \) is a positive strongly oscillatory coefficient and if \( g(t) \) increases "sufficiently fast," then any solution of (2.3) that exists on a ray \((a, \infty)\) is oscillatory. In order to avoid the assumption that \( g(t) \) is differentiable, we introduce a differentiable minorant \( h(t) \). This will allow the theorem to be applied to delay equations of the form \( Y''(t) + P(t)Y(t - \tau(t)) = 0, 0 \leq \tau(t) \leq M \), where \( \tau(t) \) is not assumed differentiable.

**Theorem 2.2.** If

(i) \( h(t) \leq g(t) \) and \( 0 < k \leq h'(t) \leq 1 \),

(ii) there exists \( M > 0 \) such that \( y \geq M \) implies

\[
\liminf_{|w| \to \infty} \frac{f(y, w)}{w} \geq \epsilon > 0,
\]

(iii) \( P(t) \geq 0 \) and \( \limsup_{x \to \infty} x \int_{x}^{\infty} P(t) \, dt = \infty \),

then all solutions of (2.3) existing on \((a, \infty)\) are oscillatory.

**Proof.** Assume the contrary. Then (2.3) has a nonoscillatory solution \( Y(t) \). Assume that \( Y(t) > 0 \) for large \( t \) (the case \( Y(t) < 0 \) can be treated similarly). It is easy to verify that \( Y'(t) > 0 \) for large \( t \). Let \( W(t) = Y'(t)/Y(h(t)) \). Then \( W(t) \) satisfies

\[
W'(t) = -P(t)\frac{f(Y(t), Y(g(t)))}{Y(h(t))} - \frac{Y'(h(t))}{Y(h(t))} h'(t)W(t).
\]

Since \( Y'(t) > 0 \) for large \( t \), \( \lim_{t \to \infty} Y(t) \) exists either as a finite or infinite limit. If \( \lim_{t \to \infty} Y(t) = b \) is finite, then

\[
\lim_{t \to \infty} \frac{f(Y(t), Y(g(t)))}{Y(g(t))} = \frac{f(b, b)}{b} > 0.
\]

If \( \lim_{t \to \infty} Y(t) = \infty \), then by (ii) we have that

\[
\frac{f(Y(t), Y(g(t)))}{Y(g(t))} \geq \epsilon > 0
\]

for large \( t \). In either case, we have that (2.4) holds for sufficiently large \( t \).
Now since $Y(t)$ is increasing for large $t$ we have that

$$P(t) \frac{f(Y(t), Y(g(t)))}{Y(h(t))} \geq P(t) \frac{f(Y(t), Y(g(t)))}{Y(g(t))} \geq \varepsilon P(t)$$

and also that

$$\frac{Y''(h(t))}{Y(h(t))} h'(t) W(t) \geq k W^2(t).$$

Therefore $W'(t) \leq -\varepsilon P(t) - k W(t)$ and if we define $R(t) = k W(t)$, we obtain $R'(t) + R^2(t) + k \varepsilon P(t) \leq 0$. However, by a well-known theorem of Wintner [7] this implies that the equation $Y''(t) + k \varepsilon P(t) Y(t) = 0$ is nonoscillatory. But this contradicts the fact that (iii) implies $P(t)$ is a strongly oscillatory coefficient.

In Theorem 2.2 we made an assumption on the rate of increase of $g(t)$. The equation

$$Y''(t) + \frac{1}{4}(t^3 \ln t)^{-1/2} Y(\ln t) = 0 \quad \text{for } 1 < t$$

demonstrates that the assumption is necessary. This equation has the nonoscillatory solution $Y(t) = t^{1/2}$ even though $P(t) = \frac{1}{4}(t^3 \ln t)^{-1/2}$ is a strongly oscillatory coefficient.

**References**


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