SEMI-LOCAL-CONNECTEDNESS AND CUT POINTS IN METRIC CONTINUA

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Abstract. In the first section of this paper, the notion of a space being rational at a point is generalized to what is here called quasi-rational at a point. It is shown that a compact metric continuum which is quasi-rational at each point of a dense subset of an open set is both connected im kleinen and semi-locally-connected on a dense subset of that open set. In the second section a $G_δ$ set is constructed such that every point in the $G_δ$ at which the space is not semi-locally-connected is a cut point. A condition is given for this $G_δ$ set to be dense. This condition, in addition to requiring that the space be not semi-locally-connected at any point of a dense $G_δ$ set, gives a sufficient condition for the space to contain a $G_δ$ set of cut points. The condition generalizes that given by Grace.

1. Throughout this paper $M$ will be taken to be a compact metric continuum. Many of the lemmas, however, can be proven with less hypotheses. Lemma 2, for example, requires only that the sets $P_x$ (defined below) be subcontinua of $M$. Compact Hausdorff is sufficient for this to happen [4].

Let $x$, $y$, and $z$ be points of $M$ (not necessarily distinct). The point $x$ cuts between $y$ and $z$ in $M$ when every subcontinuum of $M$ which contains both $y$ and $z$ must also contain $x$. The point $x$ is a cut point of $M$ when $x$ cuts between two points distinct from $x$. $M$ is said to be aposyndetic (semi-locally-connected) at $x$ with respect to $y$ if and only if there is a subcontinuum of $M$ with $x$ ($y$) in its interior that does not contain $y$ ($x$). $M$ is aposyndetic (semi-locally-connected) at $x$ when it is aposyndetic (semi-locally-connected) at $x$ with respect to every other point. Finally, $M$ is connected im kleinen at $x$ when each neighborhood of $x$ contains a closed neighborhood of $x$ which is also connected. One should note that when $M$ is connected im kleinen at a point, it is also aposyndetic at that point. For $x \in M$, $P_x$ denotes $\{y \in M \mid M$ is not aposyndetic at $y$ with respect to $x\}$, and for $T \subseteq M$, $P_T$ denotes $\bigcap \{H \mid T \subseteq H^0$ and $H$ is a
subcontinuum of \( M \}). Clearly \( M \) is semi-locally-connected at \( x \) if and only if \( P_x = \{x\} \). As noted above, \( P_x \) is a subcontinuum of \( M \).

**Lemma 1.1.** If \( T \) is a subcontinuum of \( M \) and \( x \) is a point of \( M \), then \( x \in P'_T \) if and only if \( P_x \cap T \neq \emptyset \).

**Proof.** Suppose \( x \in P'_T \). If \( P_x \cap T = \emptyset \), then \( M \) is aposyndetic at every point of \( T \) with respect to \( x \). By the definition of aposyndetic and the compactness of \( T \) we see that \( T \) can be covered by the interior of a finite number of continua \( A_1, \ldots, A_n \), where each \( A_i \) meets \( T \) and does not contain \( x \). Now \( T \cup \bigcup_{i=1}^n A_i \) is a subcontinuum of \( M \) containing \( T \) in its interior, and hence \( x \in P'_T \subseteq T \cup \bigcup_{i=1}^n A_i \). Since \( x \notin A_i \), we have \( x \in T \), and so \( x \in P_x \cap T = \emptyset \). This contradiction shows \( P_x \cap T \neq \emptyset \).

Conversely, suppose that for some \( x \in M \), \( P_x \cap T \neq \emptyset \), say \( y \in P_x \cap T \). Let \( T \subseteq H^0 \) where \( H \) is a subcontinuum of \( M \). Then \( y \in H^0 \). Since \( y \in P_x \), \( M \) is not aposyndetic at \( y \) with respect to \( x \). It follows that \( x \in H \). Thus \( x \in P'_T \). □

**Lemma 1.2.** If \( T \) is a subcontinuum of \( M \), then for \( z \in (P'_T)^0 \) and \( y \in M - T \) we have \( z \in P_v \) implies \( y \in P_z \).

**Proof.** Suppose \( y \notin P_z \) and \( z \in P_v \). Then there is a continuum \( H \) containing \( y \) in its interior which does not contain \( z \). Let \( U \) be an open neighborhood of \( y \) in \( H^0 \cap (M - T) \), and let \( L \) be the component of \( M - U \) containing \( T \). Suppose \( x \in (M - H) \cap P'_T \). Then in particular \( P_x \cap U \neq \emptyset \). If \( x \notin L \), then \( L \) is a proper subset of \( P_x \cup L \). Hence \( P_x \cap U \neq \emptyset \). Let \( s \in P_x \cap U \), then \( s \in H^0 \) and thus \( x \in H \). With this contradiction we conclude that \( (M - H) \cap P'_T \subseteq L \). Thus \( z \in (M - H) \cap (P'_T)^0 \subseteq L^0 \) and, of course, \( z \in P_v \). This implies \( y \in L \) which contradicts the fact that \( y \in U \subseteq M - L \). □

**Lemma 1.3.** Let \( V \) be an open point set of \( M \). \( M \) is semi-locally-connected on a dense subset of \( V \) if and only if for each open point set \( W \) in \( V \), there is a finite number of continua covering \( \partial W \) but not all of \( W \).

**Proof.** That this condition is necessary is immediate, for if \( W \) is an open point set of \( V \), then there is a point \( x \in W \) at which \( M \) is semi-locally-connected. Thus \( M \) is aposyndetic at each point of \( \partial W \) with respect to \( x \). Since \( \partial W \) is compact we can conclude there is a finite number of continua covering \( \partial W \) with their interior but not containing \( x \).

Conversely let \( W \) be any open point set of \( V \). We will find a point \( x \in W \) at which \( M \) is semi-locally-connected. By the hypothesis we can...
choose open point sets $W_i$ and continua $H^i_1, \cdots, H^i_n$, such that

1. $W_i \subseteq W,$
2. $\partial W_i \subseteq \bigcup_{j=1}^{n_i} H^j_i,$
3. $\overline{W}_{i+1} \subseteq W_i - \bigcup_{j=1}^{n_i} H^j_i,$
4. $x, y \in W_i$ implies $d(x, y) \leq 1/i.$

Let $x \in \bigcap W_i$. For $y \neq x$, choose $k$ such that $y \notin N_{2/k}(x)$ ($N_r(x)$ is the open ball with center $x$ and radius $r$). Then $x \in (W_k - \bigcup_{j=1}^{n_k} H^j_k) = U_k$ and $y \in M - U$. Now each component of $M - U$ meets $\partial U$ which is in $\bigcup_{j=1}^{n_k} H^j_k$. Thus $M - U$ has only a finite number of components (each component of $M - U$ contains at least one $H^j_k$). Since $y \in M - U$, $y$ is in the interior of the component of $M - U$ containing $y$. Since this component does not contain $x$, $M$ is semi-locally-connected at $x$ with respect to $y$. It follows that $M$ is semi-locally-connected at $x$ which completes the proof. □

$M$ is said to be quasi-rational at $x$ if and only if for each open neighborhood $W$ of $x$ there is an open neighborhood $U$ of $x$ in $W$ such that $W - U$ contains a closed set which is a countable union of continua and which separates $U$ from $M - W$.

**Lemma 1.4.** If $M$ is quasi-rational on a dense subset of an open point set $V$ of $M$, then $M$ is connected im kleinen on a dense subset of $V$.

**Proof.** Let $W$ be an open point set in $V$. We will show $W$ contains a point at which $M$ is connected im kleinen. $W$ contains a point at which $M$ is quasi-rational. Thus there is an open point set $U$ and continua $T_1, T_2, \cdots$ such that $\bigcup T_i \subseteq W - U$ is closed and separates $U$ from $M - W$. Since each component of $M - U$ meets some $T_i$, we see $U$ is covered by a countable number of continua in $W$. One of these continua must contain an open subset of $U$. The above proof procedure allows us to verify that there are continua $H_1, H_2, \cdots$ in $W$ such that for each positive integer $i$, $H_{i+1} \subseteq H^i_i$ and the diameter of $H_i$ is $\leq 1/i$. Let $x \in \bigcap H_i$. Then since $x \in H^0_i$ for each $i$ and for each neighborhood $G$ of $x$ there exists an integer $i$ such that $H_i$ is contained in $G$, $M$ is connected im kleinen at $x$. □

**Theorem 1.1.** If $M$ is quasi-rational on a dense subset of an open point set $V$ then $M$ is semi-locally-connected on a dense subset of $V$.

**Proof.** Suppose not. By Lemma 1.3 there is an open point set $W$ in $V$ such that if $\partial W$ is covered by a finite number of continua, then they cover all of $W$. This implies in particular that $P_x \cap \partial W \neq \emptyset$ for all $x \in W$. Now let $U$ be an open point set in $W$ which is separated from $M - W$ by a countable union of continua, $\bigcup T_i$, which is a closed subset of $W - U$. Since $P_x$ is connected and $P_x \cap \partial W \neq \emptyset$ for $x \in W$, for $x \in U$ we have $P_x \cap \bigcup T_i \neq \emptyset$. Let $K_i = \{x \in U \mid P_x \cap T_i \neq \emptyset\}$. 

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It is easily seen that $K_i$ is a closed subset of $U$ (relative topology) for each $i$ [4, Theorem 1]. Since the $K_i$‘s form a countable cover of $U$, it follows that for some $i$, $K_i^0 \neq \emptyset$. By Lemma 1.1 we see $K_i^0 \subseteq P_{T_i} \cap (M - T_i)$. By Lemma 1.4 there is a point $x \in U \cap (P_{T_i})^0$ at which $M$ is apopsyndetic. Since $P_x \cap \partial W \neq \emptyset$ and $P_x$ is connected, there is a point $y \in U \cap (P_{T_i})^0 \cap P_x$ different from $x$. By Lemma 1.2, $x \in P_y$ which contradicts the fact that $M$ is apopsyndetic at $x$ with respect to $y$. □

2. In the following a $G_\delta$ set is constructed such that every point in the $G_\delta$ at which $M$ is not semi-locally-connected is a cut point. Then it is proven that under certain conditions this $G_\delta$ is dense. In this section $y$ is a fixed point of $M$. $C(x, i)$ is used to denote the component of $M - N_{1/i}(x)$ containing $y$, and when it is used it is assumed that $y \in M - N_{1/i}(x)$. Let $G_n = \{z \in M \mid$ there is a point $x \in M$ and integers $i, j$ such that $d(x, z) < 1/n, i > n,$ and $C(x, i) \subseteq C(z, j)^0\}$ and let $G = \bigcap G_n$.

**Lemma 2.1.** $G_n$ is an open set for each $n$.

**Proof.** Let $z \in G_n$. There is a point $x$ of $M$ and integers $i, j$ such that $d(x, z) < 1/n$, and $C(x, i) \subseteq C(z, j)^0$. $N_{1/n}(x) \cap N_{1/j}(z)$ is a neighborhood of $z$. For $s \in N_{1/n}(x) \cap N_{1/j}(z)$ we have $d(x, z) < 1/n$, and we can find a $k$ so that $N_{1/k}(s) \subseteq N_{1/j}(z)$. Hence $C(z, j) \subseteq C(s, k)$. It follows that $C(s, i) \subseteq C(z, j)^0 \subseteq C(s, k)^0$, and thus $s \in G_n$. □

**Lemma 2.2.** If $z \in G$ and $z$ is not a cut point, then $M$ is semi-locally-connected at $z$.

**Proof.** Suppose $z \in G$ is not a cut point. For each positive integer $n$ there exists a point $x_n$ and integers $i_n, j_n$ such that $d(x_n, z) < 1/n, i_n > n$ and $C(x_n, i_n) \subseteq C(z, j_n)^0$. Now $C(z, j_n)$ is a continuum containing $z$, so $P_x \subseteq \bigcap_n (M - C(z, j_n)^0) \subseteq \bigcap_n (M - C(x_n, i_n))$. Suppose $s \in \bigcap_n (M - C(x_n, i_n))$ and $s \neq z$. Let $H$ be a subcontinuum of $M$ joining $s$ to $y$ and missing $z$. Choose $k$ large enough so that $N_{1/k}(z) \cap H = \emptyset$. Then $H \subseteq C(z, k)$. Also choose $p$ large enough so that $N_{1/p}(x_p) \subseteq N_{1/k}(z)$. Then $C(z, k) \subseteq C(x_p, p)$ and hence $s \in C(x_p, p)$. This contradiction shows $\bigcap (M - C(x_n, i_n)) = \{z\}$. Thus $P_z \subseteq \{z\}$ and $M$ is semi-locally-connected at $z$. □

**Theorem 2.1.** Let $V$ be an open set in $M$. Suppose for all continua $T$ containing $y$ we have that $(P_T)^0 \cap (V - T) = \emptyset$, then $V \cap G$ is dense in $V$.

**Proof.** Suppose $W \subseteq V - G$ is an open point set of $M$. Let $x_1 \in W$. Choose $i_1 > 1$ such that $N_{1/i_1}(x_1) \subseteq W$ (with no loss of generality $y \notin W$).

If there is an $x \in N_{1/i_1}(x_1)$ such that, for some $j$, $C(x_1, i_1) \subseteq C(x, j)^0$ then
we let $x_2 = x$ and $i_2 = \max (j, 2, k)$ where $k$ is such that

$$\overline{N}_{i_k}(x) \subseteq N_{1/i_k}(x).$$

Suppose for each positive integer $n$ there exists a point $x_n$ and an integer $i_n \geq n$ such that

$$C(x_n, i_n) \subseteq C(x_{n+1}, i_{n+1})^0 \quad \text{and} \quad \overline{N}_{1/i_{n+1}}(x_{n+1}) \subseteq N_{1/i_n}(x_n).$$

Let $z \in \bigcap N_{1/i_n}(x_n)$. Then $x_1, x_2, x_3, \ldots$ converges to $z$. Since $z \in N_{1/i_{n+1}}(x_{n+1})$, for each positive integer $n$ there is a $j_n$ such that $N_{1/j_n}(z) \subseteq N_{1/i_{n+1}}(x_{n+1})$. Hence we conclude that $C(x_n, i_n) \subseteq C(x_{n+1}, i_{n+1})^0 \subseteq C(z, j_n)^0$. Since $d(x_n, z) < 1/n$ it follows that $z \in G_n$ for all $n$. But this says $z \in G \cap W$. We conclude that there must be an $n$ such that $x \in N_{1/i_n}(x_n)$ implies $C(x_n, i_n) \nsubseteq C(x, j)^0$ for all $j$. Let $T = C(x_n, i_n)$. $T$ is a subcontinuum of $M$ containing $y$. Let $s \in N_{1/i_n}(x_n)$ and $T \subseteq H^0$ where $H$ is a subcontinuum of $M$. If $s \notin H$, then there is a $j$ such that $N_{1/j}(s) \subseteq M - H$. Hence $H \subseteq C(s, j)$. This says $C(x_n, i_n) = T \subseteq H^0 \subseteq C(s, j)^0$ which is a contradiction. Therefore $N_{1/i_n}(x_n) \subseteq P_T \cap (V - T)$, contradicting the fact that

$$(P_T)^0 \cap (V - T) = \emptyset. \quad \square$$

**Corollary 2.1.** If $M$ is not semi-locally-connected at any point of a dense $G_\delta$ subset of an open point set $V$, and if for any subcontinuum $T$ of $M$ containing $y$ we have $(P_T)^0 \cap (V - T) = \emptyset$. Then $V$ contains a dense $G_\delta$ set of cut points.

**Corollary 2.2 (Grace [2]).** Suppose $V$ is an open set of $M$ which contains a dense $G_\delta$ set $G$ such that given any point $x$ in $G$, $M$ is locally peripherally aposyndetic at $x$ and $M$ is not semi-locally-connected at $x$. Then $V$ contains a dense $G_\delta$ set of cut points.

$(M$ is locally peripherally aposyndetic at $x$ when for $x \in U$, $U$ open, there is an open set $W$ such that $x \in W \subseteq U$ and $M$ is aposyndetic at $x$ with respect to each point of $\partial W$.)

**Proof.** If $V$ does not contain a dense $G_\delta$ set of cut points, then by Corollary 2.1 there is a continuum $T$ such that $(P_T)^0 \cap (V - T) \neq \emptyset$. Let $x \in (P_T)^0 \cap (V - T)$ be a point at which $M$ is both locally peripherally aposyndetic and semi-locally-connected. Since $M$ is not semi-locally-connected at $x$, there is an open set $W$ such that $x \in W \subseteq (P_T)^0 \cap (V - T)$ and $P_x \cap (M - W) \neq \emptyset$. Let $U$ be open such that $x \in U \subseteq W$ and $M$ is aposyndetic at $x$ with respect to each point of $\partial U$. Since $P_x$ is a continuum there is a $z \in P_x \cap \partial U$. By Lemma 1.2, $x \in P_z$. But this says $M$ is not aposyndetic at $x$ with respect to $z$ and since $z \in \partial U$ we have a contradiction. \(\square\)
Jones [4, Theorem 15] has shown that a compact metric continuum $M$ which is not semi-locally-connected at any of its points contains a dense set of cut points. Grace [1] posed the question whether a space $M$ has a $G_δ$ set of cut points. In particular, this would imply that the cardinality of the collection of cut points is $c$. Hagopian [3, Theorem 4] has shown that the latter must happen: If a compact metric continuum $M$ is not semi-locally-connected at any point of a $G_δ$ subset which is dense in $M$ then the set of cut points in each open point set has cardinality $c$.

Suppose $V$ is open in $M$ and $M$ is not semi-locally-connected at any point of a dense $G_δ$ subset $K$ of $V$. Let $V = V \cap (\bigcup \{P_2^g - T \mid T \text{ is a subcontinuum of } M\})$ and let $V_2 = (V - V_1)^0$. By Corollary 2.1, $V_2$ contains a dense $G_δ$ set of cut points. Although we cannot show that $V_1$ contains a dense $G_δ$ set of cut points (which would answer Grace’s question), we can strengthen Hagopian’s result by proving that when $V_1 \neq \emptyset$, $V_1$ contains a nondegenerate continuum whose points are cut points. Assume $V_1 \neq \emptyset$.

**Theorem 2.2.** $V_1$ contains a dense $G_δ$ set $J$ such that for each $x \in J$ there is a nondegenerate subcontinuum $H$ of $M$ containing $x$ such that each point of $H$ cuts $x$ from $y$.

**Proof.** As a special case of Grace’s Theorem 2 [2] we have $M$ contains a dense $G_δ$ set $I$ such that if $x \in I \cap P_2$ then $z$ cuts $x$ from $y$. Let $J = V_1 \cap I$. For $x \in J$ there is a subcontinuum $T$ of $M$ such that $x \in P_2^g - T$. Let $H$ be any nondegenerate subcontinuum of $P_2$ in $P_2^g$ containing $x$. By Lemma 1.2, $z \in H$ implies $x \in P_2^g$. Since $x \in I$, $z$ cuts $x$ from $y$.

By choosing a nondegenerate subcontinuum $K$ of $H$ (the $H$ of Theorem 2.2) which is contained in $V - \{x, y\}$, we have that each point of $K$ is a cut point.

**References**


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