MODULES OVER THE ENDOMORPHISM RING OF A FINITELY GENERATED PROJECTIVE MODULE

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Abstract. Let PR be a projective module with trace ideal T. An R-module XR is T-accessible if XT = X. If PR is finitely generated projective and C is the R-endomorphism ring of P, such that cPR, then for XR, Hom (PR, XR)c is artinian (noetherian) if and only if XR satisfies the minimum (maximum) condition on T-accessible submodules. Further, if XR is T-accessible then Hom (PR, XR)c is finitely generated if and only if XR is finitely generated.

The purpose of the present paper is to investigate cHom (cPR, XR), where PR is a finitely generated projective R-module and C = End (PR), the R-endomorphism ring of P, with respect to the properties of chain conditions and finite generation. Throughout this paper R is a ring with identity and all modules over R are unitary. The convention of writing module-homomorphisms on the side opposite the scalars is adopted here.

1. Preliminaries. Let PR be a finitely generated projective R-module with C = End (PR) such that cPR. The dual module of PR is (with respect to RR), PR* = Hom (cPR, rRr). It is well known, see [1], that the map PR δP, P** = Hom (PR*, RR) given by p → p, where fp = fp, for f ∈ P*, for f ∈ P* is an R-isomorphism.

Lemma 1.1. For PR finitely generated projective, C = End (PR), the map End (PR) → End (PR*) given by c → c, where fc = fc is a ring isomorphism.

Proof. The above map is nothing more than the composite of the following maps

Hom (P, P) → Hom (P, P*) → Hom (P*, P*)

where t is the natural equivalence of functors in [2, Chapter II, Exercise 4].

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Since \( \delta_P \) is an isomorphism the lemma follows. Let \( P_R \) be a projective \( R \)-module and \( T \) the trace ideal of \( P \). It is well known, e.g., see [3], that \( T \) is an idempotent two-sided ideal of \( R \) and \( PT = P \).

Some results on the trace ideal of a projective module are listed in the proposition below.

**Proposition 1.2.** Let \( P_R \) be a projective module with trace ideal \( T \), then:

(i) For \( X_R \), \( \text{Hom} (P_R, X_R) = 0 \) if and only if \( XT = 0 \).

(ii) For \( X_R \), \( XT = X \) if and only if \( X_R \) is an epimorphic image of a direct sum (coproduct) of copies of \( P_R \).

The proof of this proposition is an easy consequence of the definition and is left to the reader.

2. Main results. Throughout this section \( P_R \) denotes a finitely generated projective module, \( T \) its trace ideal, \( C = \text{End} (P_R) \), \( _RP_C^P = \text{Hom} (CP_R, RR_R) \) and, for \( X_R, X'_C = \text{Hom} (CP_R, X_R) \).

**Definition 2.1.** (a) A module \( X_R \) is \( T \)-accessible if and only if \( XT = X \).

(b) A module \( X_R \) is \( T \)-artinian (\( T \)-noetherian) if and only if \( X_R \) satisfies the minimum (maximum) condition on \( T \)-accessible submodules.

For a module \( X_R \) and \( M_C \) a submodule of \( X'_C \), \( MP \) denotes the submodule of \( X_R \) generated by the images of \( P_R \) by homomorphisms in \( M_C \). If \( Y_R \) is a submodule of \( X_R \) then \( \text{Hom} (CP_R, Y_R) = Y'_C \) can be identified with a submodule of \( X'_C \) since \( \text{Hom} (P_R, -) \) is (left) exact.

It is now clear that a submodule \( Y_R \) of \( X_R \) is \( T \)-accessible if and only if \( Y = MP \) for some submodule \( M_C \) of \( X'_C \). These facts will be used in what follows.

**Theorem 2.2.** For a module \( X_R \), the correspondence \( M_C \leftrightarrow MP \) is a one-to-one correspondence, inclusion preserving, between the submodules of \( X'_C \) and the \( T \)-accessible submodules of \( X_R \).

**Proof.** Since \( MP \) is an \( R \)-submodule of \( X_R \), the theorem will follow if \( \text{Hom} (P_R, MP_R) = M \). Clearly \( M \subseteq \text{Hom} (P_R, MP_R) \). Let \( f \in \text{Hom} (P_R, MP_R) \), then let \( MP \) be a direct sum of \( M \) copies of \( P_R \) and denote by \( \pi_m : MP \rightarrow P \) the \( m \)th projection map. Now the following diagram of \( R \)-modules and \( R \)-homomorphisms is commutative

\[
\begin{array}{ccc}
P & \xrightarrow{	ext{f'}} & M_P \\
\mu \downarrow & & \downarrow \mu \\
MP & \rightarrow & MP \\
\end{array}
\]

where \( \mu x = \sum_{m \in M} m(\pi_m x) \) and \( f' \) exists since \( P_R \) is projective.
Since $P$ is finitely generated there is a finite subset $N$ of $M$ such that 
\[ \pi_m f' p = 0 \text{ for all } p \in P \text{ and all } m \notin N. \]
Now
\[ fp = \mu f' p = \sum_{m \in N} m(\pi_m f' p) = \left( \sum_{m \in N} m(\pi_m f') \right) p \]
hence $f = \sum_{m \in N} m(\pi_m f')$. Since $\pi_m f' \in C$, $f \in M_C$ and the theorem follows.

**Corollary 1.** A module $X_R$ is T-artinian (T-noetherian) if and only if $X'_C$ is artinian (noetherian).

**Corollary 2.** If $X_R$ is T-accessible then $X_R$ is finitely generated if and only if $X'_C$ is finitely generated.

**Proof.** A well-known characterization of finitely generated modules is the following: A module $X_R$ is finitely generated if and only if every totally ordered subset, by inclusion, of proper submodules of $X_R$ has a proper submodule of $X_R$ for its union (least upper bound). By the theorem since $X_R$ is T-accessible $X'_C$ is finitely generated if and only if the union of a totally ordered set of proper T-accessible submodules of $X_R$ is a proper submodule of $X_R$, hence if $X_R$ is finitely generated so is $X'_C$.

Conversely, suppose $X'_C$ is finitely generated and \( \{ Y_i \}_{i \in I} \) is some index set, is a totally ordered subset of proper submodules of $X_R$. If $\bigcup_i Y_i = X$, then $\bigcup_i TY_i = TX = X$, hence it is sufficient to show that $\bigcup_i TY_i \neq X$. If $\bigcup_i TY_i = X$ then let $(TY_i)' = M_i \subseteq X'_C$. Since $P_R$ is finitely generated it follows that $X'_C = \text{Hom} (P_R, \bigcup_i TY_i) = \bigcup_i M_i$, a contradiction to the finite generation of $X'_C$ since $M_i \neq X'$, for if $M_i = X'$, $M_i P = TY_i = X$ and the corollary follows.

**Corollary 3.** If $0 \to U_R \to V_R \to W_R \to 0$ is exact then $V_R$ is T-artinian, respectively T-noetherian, if and only if $U_R$ and $W_R$ are T-artinian, respectively T-noetherian.

**Proof.** Since $P_R$ is projective $0 \to U'_C \to V'_C \to W'_C \to 0$ is exact and the corollary follows from an analogous result for modules and Corollary 1.

It is well known, e.g., [4], that the functors $\text{Hom} (cP_R, X_R)$ and $X \otimes_R P^*$ are naturally equivalent as functors of $X_R$, hence all previous results can be stated replacing $X'_C$ with $X \otimes_R P^*$.

In view of Lemma 1.1, $R_P C = \text{Hom} (cP_R, R_R)$, an endomorphism of $R_P^*$ is given by a unique $c \in C$, namely if $d \in \text{End} (R_P^*)$, there is a unique $c \in C$ such that $fd = fc$ for every $f \in P^*$. With this identification the following is valid.

**Corollary 4.** $R_T$ is finitely generated if and only if $cP$ is finitely generated.
Proof. \( cP \cong \text{Hom}(P^*, T) \) so by Corollary 2, the above follows. Some obvious results of the preceding are listed in the next proposition without proof.

**Proposition 2.3.** If \( P \), \( C \) are as in Theorem 2.2, then

(i) If \( P \) is artinian (noetherian), so is \( C \).

(ii) If \( P \) is artinian (noetherian), so is \( P_c \).

(iii) If \( X \) is artinian (noetherian), so is \( \text{Hom}(P, X) \).

Now will be taken up the problem of whether direct sums in the correspondence of Theorem 2.2 are preserved. Since \( P \) is finitely generated, it follows that if \( \sum_i X_i \) is a direct sum of submodules of \( X \), then \( \sum (X_i)^\prime = \sum X_i^\prime \) is a direct sum of submodule of \( X \).

For the question of whether \( \sum_i M_i \) is a direct sum whenever \( \sum_i M_i \) is a direct sum of \( R \) submodules of \( X \), the following notion will be useful.

**Definition 2.4.** For the module \( X \), \( T \) is \( X \)-faithful if \( xT \neq 0 \) for each \( x \in X \).

**Theorem 2.5.** Let \( X \) be such that \( T \) is \( X \)-faithful, then if \( \sum_i M_i \) is a direct sum of \( C \) submodules of \( X \), then \( \sum_i M_i \) is a direct sum of \( R \) submodules of \( X \).

Proof. For an index \( j \in I \),

\[
\left[ M_j \cap \left( \sum_{i \neq j} M_i \right) \right] \subseteq (M_j)^\prime \cap \left( \sum_{i \neq j} M_i \right)^\prime \subseteq M_j \cap \sum_{i \neq j} M_i = 0,
\]

where the last inclusion follows from the facts that \( (M_j)^\prime = M_j \) and since \( P \) is finitely generated,

\[
\left( \sum_{i \neq j} M_i \right)^\prime \subseteq \sum_{i \neq j} (M_i)^\prime = \sum M_i.
\]

Now by Proposition 1.1,

\[
T \left[ M_j \cap \left( \sum_{i \neq j} (M_i) \right) \right] = 0
\]

and since \( T \) is \( X \)-faithful, \( M_j \cap (\sum_{i \neq j} M_i) = 0 \) and the theorem follows.

**Corollary 1.** If \( T \) is \( X \)-faithful and \( X \) has finite Goldie dimension, see [3], \( X \) has finite Goldie dimension.

**Corollary 2.** If \( T \) is faithful with finite Goldie dimension, then \( C \) has finite Goldie dimension.
PROOF. It is sufficient to show $T$ is $P$-faithful. Since $P_R$ is a direct summand of a free $R$-module, if $xT = 0$ for some $0 \neq x \in P$, $rT = 0$ for some $0 \neq r \in R$, a contradiction, so $T$ is $P$-faithful.

REFERENCES


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