MORE ON THE SCHUR SUBGROUP

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Abstract. Let \( k \) be an abelian extension of the rational field \( \mathbb{Q} \). We show Schur’s subgroup \( S(k) \) of the Bauer group \( B(k) \) is usually of infinite index. Generators for \( p \)-torsion elements of \( S(k) \) are found when \( k \) is the cyclotomic field of \( p \)-th roots of unity.

Let \( k \) be an algebraic number field. We write \( B(k) \) for the Brauer group of \( k \), and \( B_n(k) \) for the subgroup of \( B(k) \) generated by classes of division rings of exponent \( n \). Let \( S(k) \) be the subgroup of \( B(k) \) consisting of all classes which contain a simple component of \( \mathbb{Q}[G] \), the group algebra of a finite group \( G \) over the rational field \( \mathbb{Q} \). Following [6] we call \( S(k) \) the Schur subgroup of \( k \). Let \( S_n(k) = S(k) \cap B_n(k) \). In [5] the structure of \( S_3(k) \) for \( k = \mathbb{Q}(\sqrt[3]{-3}) \) is determined. Theorem 2 of this note generalizes the results of [5].

If \( A \) is a central simple algebra over \( k \), we write \([A]\) for the corresponding class in \( B(k) \).

Theorem 1. \( S_n(k) \) is of infinite index in \( B_n(k) \) for all \( n \geq 2 \) unless \( n = 2 \) and \( k = \mathbb{Q} \). In the exceptional case \( B_2(\mathbb{Q}) = S_2(\mathbb{Q}) \).

Proof. Since \( S(k) \) is trivial unless \( k \) is the field of an irreducible character of a finite group \( G \), we may assume \( k/\mathbb{Q} \) is abelian. Assume \( k \neq \mathbb{Q} \) and \( r = [k: \mathbb{Q}] \). There are infinitely many primes of \( \mathbb{Q} \) which split completely in \( k \); let \( p_1, \cdots, p_n, \cdots \) be an infinite list of them. For each \( p_i \), let \( q_1, \cdots, q_{m_i} \) be the primes of \( k \) lying over \( p_i \). We construct \([D_1], [D_2], \cdots, [D_m], \cdots\) in \( B_n(k) \) as follows:

\[ D_m \] is the central division ring over \( k \) whose Hasse invariants satisfy:

\[
\begin{align*}
\text{inv}_{q_{m-1}^i} D_m &= \frac{1}{n}, \\
\text{inv}_{q_m^i} D_m &= -\frac{1}{n}, \\
\text{inv}_{q_i} D_m &= 0 \text{ at all other primes } q \text{ of } k.
\end{align*}
\]

The construction of the \( D_m \) is allowed by [1, Theorem 7.8]. By [2] we...

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have: \([D] \in S_n(k) \Rightarrow \) for each \(i\), \(D\) has constant index at \(g_1, \cdots, g_t\). Hence \([D_1], \cdots, [D_m], \cdots\) above represent distinct cosets of \(S_n(k)\) in \(B_n(k)\). It follows that \([B_n(k):S_n(k)] = \infty\).

If \(k = Q\), then \(S(k) = S_2(k)\) by the Brauer-Speiser theorem (see [6]).

The fact that \(B_2(Q) = S_2(Q)\) follows from [4].

Theorem 1 was also noted by Burton Fein.

Let \(p\) be a fixed odd prime. We will classify the algebras of index \(p\) in \(S(k)\), where \(k = Q(\xi_p)\) is the cyclotomic field of \(p\)th roots of unity. This generalizes Theorem 2 of [5].

If \(q\) is a prime, \(q \equiv 1 \pmod{p}\), then \(q\) splits completely in \(k = Q(\xi_p)\).

Let \(g_1, \cdots, g_{p-1}\) be the primes of \(k\) lying over \(q\). The field \(L = Q(\xi_q, \xi_p)\) is cyclic over \(Q(\xi_q)\) of degree \(q - 1\); let \(\tau\) be the generator of the Galois group of \(L\) over \(k\). Let \(H\) be the group generated by \(x, y, z\) where \(x^q = y^q = 1\), \(z\) acts on \(\langle y \rangle\) according to the Galois action of \(\tau\) on \(Q(\xi_q)\), \(z^{q-1} = x\), and \(x\) is central in \(H\). Then the cyclic algebra \(\mathfrak{A} = (k(\xi_q), \tau, \xi_p)\) is a homomorphic image of \(Q[H]\), so \([\mathfrak{A}]\) is in the Schur subgroup of \(k\). Clearly \(\mathfrak{A}^p\) is a total matrix algebra since \(\xi_p^q = 1\); so \([\mathfrak{A}]\) has order \(1\) or \(p\) in \(B(k)\). \([\mathfrak{A}]\) has order \(1 \iff \xi_p\) is a norm from \(L = k(\xi_q)\) to \(k\). We show \(\xi_p\) is not a local norm at the primes \(g_1, \cdots, g_{p-1}\) above. For convenience we fix \(g = g_1\).

The extension \(L/k\) is totally and tamely ramified at \(g\); let \(t\) be the unique prime of \(L\) lying over \(g\). If \(U_t\) (resp. \(U_g\)) denotes the units of \(L_t\) (resp. \(k_g\)) and \(U_t^1\) (resp. \(U_g^1\)) those which are \(1 \pmod{t}\) (resp. \(1 \pmod{g}\)), then as in [7, \#3] the norm induces a homomorphism:

\[
N_0: U_t/U_t^1 \to U_g/U_g^1.
\]

But \(U_t/U_t^1 \cong L_t^\ast\), the multiplicative group of the residue class field of \(L\) at \(t\). Similarly \(U_g/U_g^1 \cong k_g^\ast \cong L_g^\ast \cong Z_q^\ast\) as \(g\) is totally ramified in \(L\). Thus (1) reduces to a homomorphism:

\[
N_0: Z_q^\ast \to Z_q^\ast
\]

of cyclic groups of order \(q - 1\). By [7, Proposition 5, p. 92] we have:

\[
N_0(x) = x^{q-1} \text{ in (2). Hence the image of } N_0 \text{ is trivial, so } N_0 \text{ does not cover the image of } \xi_p; \text{ it follows that } \xi_p \text{ is not a norm.}
\]

Thus \(\mathfrak{A}\) represents an element of order \(p\) in \(S(k)\). Clearly \(\mathfrak{A}\) is split at all primes \(w, w \notin \{g_1, \cdots, g_{p-1}\}\), for each such prime is unramified from \(k\) to \(L\) and so \(\xi_p\) is a unit, hence a norm, at \(w\). By the proof of Theorem 5 of [3] we have with suitable relabelling, invariants of \(\mathfrak{A}\) of form \(1/p\), \(2/p\), \(\cdots\), \((p - 1)/p\) at \(g_1, \cdots, g_{p-1}\).

We claim \(S_p(k)\) is generated by the classes \([\mathfrak{A}]\) above. Suppose \(D\) is a central division algebra over \(k\) with \([D] \in S_p(k);\) \(D\) has exponent \(p\). If \(p\) is a rational prime, \(q \equiv 1 \pmod{p}\), and \(g_1, \cdots, g_{p-1}\) are the primes of
$k = Q(\xi_p)$ lying over $q$, we have by [2], $\text{inv}_{g_i} D = 0 \Rightarrow \text{inv}_{g_i} D = 0$, $i = 1, \cdots, p - 1$. Assume $\text{inv}_{g_i} D = a/p$, $(a, p) = 1$. Set $x = [g_1]^a \cdot [D] \in S_p(k)$; then $x$ has invariant $0$ at $g_i \Rightarrow \text{inv}_{g_i} x = 0$ for $i = 1, \cdots, p - 1 \Rightarrow \text{inv}_{g_i} D = ai/p = i(a/p)$. Thus $D$ has invariants of type $1/p$, $2/p, \cdots, (p - 1)/p$ at split primes over $q$ where $q \equiv 1 \pmod p$. We must show that $D$ has no other non-0 invariants. By appropriate multiplication in $B(k)$ as above we may assume inv$_g D = 0$ for all $q$ lying over completely split primes of $Q$.

$D$ has no non-0 invariants at primes of $k$ lying over odd rational primes by [8, Satz 10]. Also, since the index of $D$ is $p$ and $p \neq 2$, then $D$ has no non-0 invariants at primes of $k$ extending $2$ by [8, Satz 11]. We have proved:

**Theorem 2.** If $p$ is an odd prime, then the division rings $D$ with $[D] \in S_p(k), k = Q(\xi_p)$ have invariants of type $1/p, 2/p, \cdots, (p - 1)/p$ at completely split primes of $k$, and 0 everywhere else. The classes $[g_i]$ above generate $S_p(k)$.

We note that Theorem 2 has the following unusual consequence: If $p$ and $q$ are distinct odd primes, then $\xi_p$ is not a norm from $Q(\xi_p, \xi_q)$ to $Q(\xi_p) \Leftrightarrow q \equiv 1 \pmod p$.

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**References**


