ON A TAUBERIAN THEOREM OF WIENER AND PITT

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Abstract. N. Wiener and H. R. Pitt established a tauberian theorem which is "intermediate" between that of Wiener and Ikehara on one hand and a theorem of Hardy and Littlewood on the other. A new proof of the Wiener-Pitt theorem is given, using a technique of Bochner.

N. Wiener and H. R. Pitt established in [6] a tauberian theorem which is "intermediate" between that of Wiener and Ikehara on one hand and a theorem of Hardy and Littlewood on the other. Let \( \alpha \in (0, 1) \) and \( B > 0 \). Let \( C = C(\alpha, B) \) be the curve in the complex plane given by

\[
\{ \sigma + it : |t| = B\sigma^\alpha, 0 \leq \sigma < \infty \}.
\]

From a one sided boundedness condition on a function plus an \( L_1 \) hypothesis upon its Laplace transform along the curve \( C \), it is proved that the function has a small integral over certain intervals. A result of this type had been conjectured by J. Karamata and a special case (\( \alpha = \frac{1}{2} \)) was given by V. G. Avakumović [1].

The proof of Wiener and Pitt was quite intricate. Another version appears in Pitt's book [3, pp. 135-138], but that proof is valid if and only if \( \alpha \geq \frac{1}{2} \). This is so because Pitt assumes that

\[
F(u) := \int_{-\infty}^{\infty} e^{itu} \exp (-|t|^{1/\alpha}) \, dt > 0,
\]

and this inequality is valid for all real \( u \) precisely when \( \alpha \geq \frac{1}{2} \) (cf. [2] and [4]). The object of the present paper is to give a shorter proof of the theorem. We shall use a Bochner type argument, integrating along line segments \( \sigma + it, -\lambda \leq t \leq \lambda \), where we take \( \sigma = L/x, \lambda = B'\sigma^\alpha = \frac{1}{2}B'L^x x^{-\alpha} \). The number \( L \) will be chosen later and will be independent of \( x \), and \( B' = \min (B, 1) \). We formulate the theorem substantially as in [3].

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Theorem. Let \( s \) be a real measurable function that is supported in \((0, \infty)\), bounded from below, and satisfies \( \int_0^\infty |s(y)| e^{-\sigma y} \, dy < \infty \) for all positive \( \sigma \). For \( \text{Re} \omega > 0 \), define

\[
S(\omega) = \int_0^\infty e^{-\omega y} s(y) \, dy
\]

and assume that

\[
\lim_{\epsilon \to 0^+} \int |S(\omega + \epsilon) - S(\omega)| \, |d\omega| = 0,
\]

where the integration is taken over any bounded arc of \( \mathbb{C} \). Then for any \( A > 0 \),

\[
\lim_{x \to 0^+} \int_{x}^{x+Ax^2} |S(\omega + \epsilon) - S(\omega)| \, |d\omega| = 0.
\]

Since \( s \) has no support near zero, \( S(\omega) \) vanishes exponentially as \( \text{Re} \omega \to \infty \). This fact plus the \( L_1 \) limit hypothesis guarantee both the absolute integrability of \( S \) along all of \( \mathbb{C} \) and the validity of Cauchy’s formula. Thus, for \( |t| \leq \lambda \) we can write

\[
S(\sigma + it) = \frac{1}{2\pi i} \int_{\mathbb{C}} \frac{S(z)}{z - \sigma - it} \, dz.
\]

If we set

\[
h_\lambda(y) = \lambda \left( \sin \frac{\lambda y/2}{\lambda y/2} \right),
\]

we have the following familiar Parseval formula (cf. [5, p. 84])

\[
(*) \quad \int_{-\lambda}^{\lambda} (1 - \frac{|t|}{\lambda}) e^{it\sigma} S(\sigma + it) \, dt = \int_0^\infty h_\lambda(x - y) e^{-\sigma y} s(y) \, dy.
\]

Let \( \varphi(x) \) denote the left-hand side of \((*)\) (recalling that \( \sigma \) and \( \lambda \) are functions of \( x \)). We begin by establishing a type of Riemann-Lebesgue lemma.

**Lemma 1.**

\[
\lim_{x \to \infty} \varphi(x) = 0.
\]

**Proof.** Expressing \( S(\sigma + it) \) by the Cauchy formula and inverting the integration order, we have

\[
\varphi(x) = \frac{1}{2\pi i} \int_{\mathbb{C}} S(z) \left[ \int_{-\lambda}^{\lambda} \frac{(1 - \frac{|t|}{\lambda})}{z - \sigma - it} e^{it\sigma} \, dt \right] \, dz.
\]

Let \( f_{-\lambda}^\lambda \) denote the inner integral, and integrate it by parts.

\[
\left| f_{-\lambda}^\lambda \right| = \left| \int_{-\lambda}^{\lambda} \frac{e^{it\sigma}}{ix} \left( \frac{(1 - \frac{|t|}{\lambda})}{z - \sigma - it} - \frac{\text{sgn} \, t}{\lambda(z - \sigma - it)} \right) \, dt \right|.
\]

\[
\leq \frac{1}{x} \int_{-\lambda}^{\lambda} \frac{dt}{|z - \sigma - it|^2} + \frac{1}{\lambda x} \int_{-\lambda}^{\lambda} \frac{dt}{|z - \sigma - it|}.
\]
Write $z = u + iv$. If $|v| \geq \frac{3}{8}B'\sigma^a$, then
\[|z - \sigma - it| \geq |\text{Im}(z - \sigma - it)| \geq B'\sigma^a/4\]
and $\int_{-1}^{1} = o(1)$ as $x \to \infty$. If $|v| < \frac{3}{8}B'\sigma^a$, then $0 \leq u < (3B'/4)^{1/2}\sigma$ and
\[|z - \sigma - it|^2 = (u - \sigma)^2 + (v - i)^2 \geq \sigma^2 + (v - t)^2,
\]
where $\sigma^2 = 1 - \frac{(3\beta'/4)^{1/2}}{4} > 0$. Thus, in this case
\[
\left| \int_{-1}^{1} \frac{1}{x} \int_{-\infty}^{\infty} \frac{dt}{c^2\sigma^2 + t^2} + \frac{2\lambda}{\lambda x} \right| = O(1).
\]
Now express $\int_{C}$ as the integral over $C_1 = \{z \in C : |\text{Im}z| < \frac{3}{8}B'\sigma^a\}$ plus the integral over $C_2 = C - C_1$ and note that $\int_{C_1} |S(z)| |dz| \to 0$ as $1/x$ (and hence $\sigma$) tends to 0, while the same integral over $C_2$ is uniformly bounded. Thus $\varphi(x) \to 0$ as $x \to \infty$.

The next lemma shows $s$ to be bounded "on the average." It is essential for the estimation of $I_2$ and $I_4$ at the conclusion of the article that the number $\beta$ may be arbitrarily small and that the number $c_1$ not depend on $\beta$.

**Lemma 2.** There is a number $c_1$, which depends only on $s$, and for any number $\beta > 0$ there is a number $X_\beta$, which increases with $1/\beta$, such that, for all $x \geq X_\beta$,
\[
(2\beta x^a)^{-1} \int_{x-\beta x^a}^{x+\beta x^a} |s(y)| \, dy \leq c_1.
\]

**Proof.** By the hypothesis of the theorem, $s(y) \geq -k$ for some $k$ and hence $|s(y)| \leq s(y) + 2k$. We set $L = (2/B'\beta)^{1/2}$ (just for the proof of this lemma) and have $\sigma = L/x$, $\lambda = \frac{1}{2}B'\sigma^a = (\beta x^a)^{-1}$. Then
\[
2\left(\frac{\sin \frac{1}{2}}{\frac{1}{2}}\right)^2 \exp \{-\sigma x - \sigma/\lambda\} \frac{\lambda}{2} \int_{x-1/\lambda}^{x+1/\lambda} |s(y)| \, dy
\leq \int_{0}^{\infty} h_\lambda(x - y)e^{-\sigma y}s(y) + 2k \, dy = \varphi(x) + 2k \int_{0}^{\infty} h_\lambda(x - y)e^{-\sigma y} \, dy
\leq |\varphi(x)| + 2k \exp \{-\sigma x + \lambda e^L/\lambda\} \int_{|y - x| < e^{L/\lambda}} h_\lambda(x - y) \, dy
+ 2k \int_{|y - x| \geq e^{L/\lambda}} h_\lambda(x - y) \, dy
\leq |\varphi(x)| + 2k \exp \{-L + \beta Le^L x^{a-1}\} + 16ke^{-L} < 20ke^{-L}
\]
provided that $x$ is large enough so that $|\varphi(x)| \leq ke^{-L}$, $\exp \{\beta Le^L x^{a-1}\} \leq \frac{3}{8}$. Thus
\[
\frac{\lambda}{2} \int_{x-1/\lambda}^{x+1/\lambda} |s(y)| \, dy \leq \frac{20k}{8(\sin \frac{1}{2})^2} \exp \{\beta L x^{a-1}\}
\leq 12k =: c_1
\]
provided \( x \) is large enough so that \( \exp \{ \beta Lx^{x-1} \} \leq \frac{2}{\beta}(\sin \frac{1}{2})^2 \).

Since \( \beta L \) and \( e^L \to \infty \) with \( 1/\beta \) and \( \varphi \) is continuous and tending to zero, we can take \( X_\beta \) to be the infimum of all \( x \) for which each of the three inequalities is valid. \( \square \)

**Corollary 1.** There exists a number \( c_2 \), which depends only on \( s \), such that, for all \( x \geq 0 \), \( \int_0^x |s(y)| \, dy \leq c_2 x \).

**Proof.** The hypotheses of the theorem imply that \( |s| \) is locally integrable and that \( s \) is zero near the origin. Now \( x^{-1} \int_0^x |s(y)| \, dy \) is a continuous function on \((0, \infty)\) which is bounded at 0 and at \( \infty \), and thus is bounded on \((0, \infty)\). \( \square \)

**Corollary 2.** Let \( \beta > 0 \) and suppose that \( w \geq X_\beta \), where \( X_\beta \) is as in the lemma. Suppose that \( f \) is a positive monotone function on \([w, z]\) and \( \frac{1}{\beta} \leq f(x)f(x + \beta x^s) \leq 2 \) for all \( x \in [w, z] \). If \( z \geq w + \beta w^s \), then

\[
\int_w^z f(u) |s(u)| \, du \leq 4c_1 \int_w^z f(u) \, du.
\]

**Proof.** We may assume without loss of generality that \( f \) is increasing. If \( z < (w + \beta w^s) + \beta(w + \beta w^s)^s \), then

\[
\int_w^z f(u) |s(u)| \, du \leq f(z) \int_w^z |s(u)| \, du \leq 4f(w)c_1(z - w)
\]

\[
\leq 4c_1 \int_w^z f(u) \, du.
\]

In the other case define a sequence \( \{x_n\}_{n=0}^{N+1} \) by taking \( x_0 = w \), \( x_n = x_{n-1} + \beta x_{n-1}^s \), \( n = 1, 2, \cdots \), and taking \( x_{N+1} \) to be the largest number of the sequence not exceeding \( z \). Apply the above inequalities to each of the intervals \([w, x_1], [x_1, x_2], \cdots, [x_N, z]\). \( \square \)

It is now convenient to approximate the right-hand side of (*) by a convolution. We show

**Lemma 3.**

\[
\lim_{z \to \infty} \int_0^\infty h_\lambda(x - y)s(y) \, dy = 0.
\]

**Proof.** Let \( \epsilon > 0 \) be given. Write

\[
(h_\lambda \ast s)(x) = \int_0^\infty h_\lambda(x - y)s(y) \, dy = I + II,
\]

where

\[
I = e^L \int_0^\infty h_\lambda(x - y)e^{-\sigma y}s(y) \, dy = e^L \varphi(x) = o(1)
\]

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as $x \to \infty$ and
$$II = \int_0^\infty h_\lambda(x-y)s(y)\{1 - e^{L-\sigma y}\} \, dy.$$ Write
$$II = \int_0^{x/2} + \int_{x/2}^{x-vx^2} + \int_{x+vx^2}^{\infty}$$
where $v$ is a large positive number to be specified presently. By Corollary 1,
$$\left|\int_0^{x/2}\right| \leq e^L \frac{4}{\lambda (x/2)^2} \cdot c_\lambda \frac{x}{2} = \frac{16e^Lc_\lambda}{B'L^2x^{1-\sigma}} = o(1).$$
We can estimate the second and fourth integrals using Corollary 2. We first replace $h_\lambda(x-y)$ by $4/\lambda(x-y)^2 = H(y)$, and note that $H$ satisfies the monotonicity and slow growth conditions of the corollary on each of the two ranges. Thus
$$\left|\int_{x/2}^{x-vx^2}\right| + \left|\int_x^{\infty}\right| \leq 2 \cdot 4c_\lambda e^L \frac{4}{\lambda vx^2} < 2\epsilon/5$$
provided that $x$ is sufficiently large and $v > 160c_\lambda e^L/B'L^2\epsilon$. With this choice of $v$, we estimate the third integral. If $x - vx^2 \leq y \leq x + vx^2$ and $x$ is sufficiently large, then
$$|e^{L-\sigma y} - 1| \leq 2 |L - \sigma y| \leq 2Lvx^{a-1}.$$ Thus, by Lemma 2, we have
$$\left|\int_{x-vx^2}^{x+vx^2}\right| \leq 2Lvx^{a-1}\lambda \int_{x-vx^2}^{x+vx^2} |s(y)| \, dy = O(x^{a-1}\lambda x^a) = o(1).$$
Now if $x$ is sufficiently large, depending on $s$, $L$, $C$ and $\epsilon$, we have $|(h_\lambda * s)(x)| < \epsilon$. □

**Conclusion of the argument.** Let $\chi_E$ be the indicator function of the set $E$. An easy estimate shows that for any positive number $M$ we have
$$h_\lambda * \chi_{[-M,M]}(u) = 1 + O(\lambda^{-1}(M - |u|)^{-1}), \quad |u| < M,$$
$$= O(1), \quad \text{always},$$
$$= O(M\lambda^{-1}(|u| - M)^{-2}), \quad |u| > M.$$ Here the constants implied by the $O$'s are absolute. The preceding lemma implies that $((1/2M)\chi_{[-M,M]} * h_\lambda * s)(x) \to 0$ as $x \to \infty$, where $M$ may tend to $\infty$ with $x$, so long as $x \geq M$, say. We take $M = (A + \eta)x^2$, where $\eta = \ldots$.
where $\eta$ is a positive number, presently to be specified, and write

\[
(x_{[-M,M]} * h_{\lambda} * s)(x) = \int_{-\infty}^{\infty} s(x - y)(x_{[-M,M]} * h_{\lambda})(y) \, dy
\]

\[
= \int_{-\infty}^{-(A+2\eta)x^2} + \int_{-A x^2}^{A x^2} + \int_{A x^2}^{(A+2\eta)x^2} + \int_{(A+2\eta)x^2}^{\infty}
\]

\[
\int_{-\infty}^{-(A+2\eta)x^2} + \int_{(A+2\eta)x^2}^{\infty}
\]

\[
\int_{-A x^2}^{A x^2} + \int_{A x^2}^{(A+2\eta)x^2} + \int_{(A+2\eta)x^2}^{\infty}
\]

\[
= \sum_{j=1}^{6} I_j,
\]

say.

Now

\[
I_3 = \int_{x - A x^2}^{x + A x^2} s(y) \, dy + O\left(\frac{c_1 A x^2}{\lambda x^2}\right).
\]

\[
I_2 + I_4 = O(c_1 \eta x^3), \quad I_1 + I_5 = O(c_1 M / \lambda x^3), \quad \text{by Corollary 2, and}
\]

\[
I_6 = O(M c_2 / \lambda x).
\]

The four $O$'s are absolute and the estimates are valid for all sufficiently large $x$. Thus we have

\[
\left| \frac{1}{2 A x^2} \int_{x - A x^2}^{x + A x^2} s(y) \, dy \right| \leq c_3 \left( \frac{c_1}{\eta B'L^*} + \frac{c_1 \eta}{A} + \frac{c_1 (A + \eta)}{A B'L^* \eta} + \frac{c_2 (A + \eta) x^{x-1}}{A B'L^*} + o(1) \right),
\]

where $c_3$ is absolute.

Given $\epsilon$ in $(0, 1)$, take $\eta$ so small that $c_3 c_1 / A < \epsilon / 5$. Next, take $L$ sufficiently large that

\[
(A + \eta) c_1 c_3 / (A B'L^* \eta) < \epsilon / 5.
\]

Then take $x$ so large that all the preceding inequalities are valid and

\[
c_3 c_2 (A + \eta) x^{x-1} / (A B'L^*) + c_2 o(1) < 2 \epsilon / 5.
\]

We conclude then that

\[
\left| \frac{1}{2 A x^2} \int_{x - A x^2}^{x + A x^2} s(y) \, dy \right| < \epsilon
\]

for all sufficiently large $x$, and the proof of the theorem is complete. □

Possibly the $L_1$ limit in the hypothesis of the theorem can be relaxed to the $L_1$ bound $\int_C |S(\omega)| \, d\omega < \infty$. One would then have to show (if possible) that the singularity of $S$ at zero, when approached from "within" $C$, was sufficiently weak to permit the application of Cauchy's theorem.
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