SHORTER NOTES

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A HOLOMORPHIC FUNCTION HAVING A DISCONTINUOUS INVERSE

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Abstract. An example is given of a function / which is holomorphic in the open unit ball of $I^\infty$ and which extends in a natural way to the closed unit ball. The extended function gives a one-to-one correspondence of the closed ball with itself but the inverse function fails to be continuous on the image of the open ball under the map $f$.

Let $X$ and $Y$ be Banach spaces and let $D$ be open $D \subset X$. It is unknown whether the assumption: $f: D \rightarrow Y$ is an injective holomorphic map (Fréchet differentiable [1, Chapters 3 and 26]) of $D$ onto an open set $f(D) \subset Y$ implies $f^{-1}$ is holomorphic. In this note we give an example which is related to this problem.

Let $a=(a_1, a_2, \cdot \cdot \cdot) \in l^\infty$ and assume $|a_k|<1$ for each $k$. Let $B$ and $\bar{B}$ be the open and closed unit ball respectively in $l^\infty$. Define $f: B \rightarrow \bar{B}$ by $f_a(x)=(w_1, w_2, \cdot \cdot \cdot)$ where $w_k=(x_k-a_k)/(1-\overline{a_k}x_k)$, $k=1, 2, \cdot \cdot \cdot$. Clearly $f_a$ extends to $\bar{B}$ to give a one-to-one correspondence of $B$ with $B$. Also, $Df_a(x)(y) = \frac{1-|a_1|^2}{(1-\overline{a_1}x_1)^2} y_1, \frac{1-|a_2|^2}{(1-\overline{a_2}x_2)^2} y_2, \cdot \cdot \cdot$ so that $Df_a(x)$ is a bounded linear map for each $x \in B$ and $f_a$ is holomorphic in $B$.

Now choose $a_k=k/(k+1)$ (any choice of $a_k$ satisfying $|a_k|<1$ and $\sup|a_k|=1$ will do). Then $f_a(0)=-a$ and $\|f_a(0)\|=1$. By the maximum principle [1, Theorem 3.13.1], $\|f_a(x)\|=1$ for all $x \in B$. If $|x_k|=1$ for some $k$ then $\|f_a(x)\|=1$ and if for some subsequence $\{x_k\}$ we have $|x_k-a_k| \geq \epsilon>0$ then $\|f_a(x)\|=1$. Hence $\|f_a(x)\|<1$ implies $|x_k|<1$ for all $k$ and

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\[ \lim_{k \to \infty} x_k = 1. \] However, \( f_a \) cannot be continuous at such an \( x \) for we may choose \( y \) so that \( y_k = x_k \) if \( k \neq k_0 \) and \( y_{k_0} = 1 \) so that \( \| y - x \| = |1 - x_{k_0}| \) which can be made arbitrarily small while \( \| f_a(y) - f_a(x) \| \geq 1 - \| f_a(x) \| \) which is positive and constant. We also find \( f_a^{-1}(x) = f_{-a}(x) \) so \( f_a \) is discontinuous on \( f_a^{-1}(B) \) and \( f_a^{-1} \) is discontinuous on \( f_{-a}(B) \).

**References**


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