A PROPERTY OF ARITHMETIC SETS

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Abstract. We shall show that every nonempty countable arithmetic subset of \( \mathbb{N}^\mathbb{N} \) contains at least one element \( a \) such that the singleton \( \{a\} \) itself is arithmetic. The proof is carried out by using a method in classical descriptive set theory.

It is known that (*) if no member of a nonempty \( \Sigma^1_1 \) set \( E \) is hyperarithmetic then \( E \) contains a perfect subset. (In this note, sets mean subsets of \( \mathbb{N}^\mathbb{N} \)—the set of all 1-place number-theoretic functions which we identify with Baire zero-space.) In fact, every \( \Sigma^1_1 \) set with a nonhyperarithmetic element contains a perfect subset. (See, e.g., Harrison [2, Theorem 2.12] and Mathias [4, T3200].) In what follows, we shall show that an arithmetic counterpart of the proposition (*) holds true:

Theorem 1. If no member of a nonempty arithmetic set \( A \) is an arithmetic singleton, then \( A \) contains a perfect subset.

It is evident that one can not replace "arithmetic singleton" by "arithmetic element" in our theorem.

T. G. McLaughlin has asked the following question (unpublished): Let \( A \) be a nonempty countable arithmetic set. Then, must some member of \( A \) be an arithmetic singleton? Now we can obtain an affirmative answer to this question as a direct corollary of our theorem, thus:

Corollary 2. If \( A \) is a nonempty countable arithmetic set, then \( A \) contains at least one arithmetic singleton.

Since every uncountable arithmetic set (in fact, every classical uncountable analytic set) contains a perfect subset, Corollary 2 is equivalent to Theorem 1. I do not know whether every member of a countable arithmetic set is an arithmetic singleton. This is also a problem presented by McLaughlin.

Proof of Theorem 1. We shall illustrate for the case that \( A \) is a \( \Pi^0_5 \) set. Proof is analogous for the other cases. Note that if \( A \) is a \( \Sigma^0_{n+1} \) set then we can reduce it to the case of \( \Pi^0_n \).

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Now let \( A \) be a set defined by
\[
A = \{ x \in \mathbb{N}^* \mid (\forall x_0)(\exists y_0)(\forall x_1)(\exists y_1) R(x, x_0, x_1, y_0, y_1) \}
\]
where \( R \) is \( \Pi^0_1 \). Then we have
\[
\alpha \in A \iff (\exists \beta_0)(\exists \beta_1)(\forall x_0)(\forall x_1) R(x, x_0, x_1, \beta_0(x_0), \beta_1(x_0, x_1))
\]
\[
\iff (\exists \beta)(\forall x) R(x, (x)_0, (x)_1, \beta((x)_0), \beta((x)_0, (x)_1, 1)),
\]
where \( \langle a_0, a_1, \cdots, a_k \rangle = p_0^{a_0} \cdot p_1^{a_1} \cdots p_k^{a_k} \) and \( p_i \) is the \( (i+1) \)-st prime number. (For notations used in this note, we mostly borrow from Kleene [3].) Let \( R' \) and \( R'' \) be predicates defined as follows:
\[
R'(x, s) \iff [\text{Seq}(s) \land \text{Lh}(s) = 0((\text{Lh}(s))_0, (\text{Lh}(s))_1, 2) \rightarrow R(x, (\text{Lh}(s))_0, (\text{Lh}(s))_1, \exp(s, ((\text{Lh}(s))_0, \exp(s, ((\text{Lh}(s))_0, 1), \exp(s, ((\text{Lh}(s))_0, (\text{Lh}(s))_1, 1)) - 1),
\]
where \( \exp(s, i) = (s)_i \). And
\[
R''(x, s) \iff (\forall i) \text{Seq}(s) R'(x, rstr(s, i)),
\]
where
\[
rstr(s, i) = \prod_{k < i} p_k^{s_k}, \quad \text{if Seq}(s) \land i \leq \text{Lh}(s),
\]
1, \quad \text{otherwise}.
\]
Then \( R'' \) has the following properties:
1. \( \alpha \in A \iff (\exists \beta)(\forall x) R''(x, \tilde{\beta}(x)) \),
2. \( R'' \) is \( \Pi^0_4 \) and hence for each sequence number \( s \), the set \( E_s = \{ x \mid R''(x, s) \} \) is a closed set, and
3. the Souslin system \( \mathcal{S} = \{ E_s \mid \text{Seq}(s) \} \) is monotonic; that is, for all \( \beta \) and \( x \), \( E_{\beta(x+1)} \subseteq E_{\beta(x)} \).

Now, as is usual with classical descriptive set theory, for a given sequence number \( \tilde{\gamma}(m) \), we shall define a set \( A^{\tilde{\gamma}(m)} \) as follows:
\[
(4) \quad \alpha \in A^{\tilde{\gamma}(m)} \iff (\exists \beta)(\forall x) R''(x, \tilde{\gamma}(m) \ast \tilde{\beta}(x)).
\]
Then we have
\[
\alpha \in A^{\tilde{\gamma}(m)} \iff (\exists \beta)(\forall x)(\forall i)_{i \leq m + x} R'(x, rstr((m)_0, (m)_1, i))
\]
\[
\iff (\forall i) i \leq m R'(x, (m)_i) \land (\exists \beta)(\forall i) R'(x, (m)_i \ast \tilde{\beta}(i))
\]
\[
\iff (\forall i) i \leq m [i = ((m)_0, (m)_1, 2) \rightarrow R(x, (m)_0, (m)_1, (m + i)_0, (m + i)_1, 1)]
\]
\[
\land (\exists \beta)(\forall i)[m + i = ((m + i)_0, (m + i)_1, 2) \rightarrow ((m + i)_0, (m + i)_1, 1) < m \rightarrow R(x, (m + i)_0, (m + i)_1, (m + i)_0, (m + i)_1, 1))]
\]
\[ \{(m + i)_0 \leq m \land \{ (m + i)_0, (m + i)_1, 1 \} \geq m \rightarrow R(\alpha, (m + i)_0, (m + i)_1, \gamma((m + i)_0)) \land \beta((m + i)_0, (m + i)_1, 1 - m)) \} \land \{(m + i)_0 \geq m \rightarrow R(\alpha, (m + i)_0, (m + i)_1, \beta((m + i)_0 - m), \beta((m + i)_0, (m + i)_1, 1 - m)) \} \]

The second member of the outermost conjunction in the latter formula is equivalent to

\[ (3 \exists \beta_0)(\exists \beta_1)(\forall x_0)(\forall x_1) \{(x_0, x_1, 1) \leq m \rightarrow R(\alpha, x_0, x_1, \gamma((x_0)), \gamma((x_0, x_1, 1))) \land \{(x_0) \leq m \rightarrow R(\alpha, x_0, x_1, \beta_1((x_0, x_1, 1) - m)) \land \{(x_0) \geq m \rightarrow R(\alpha, x_0, x_1, \beta_1((x_0) - m), \beta_1((x_0, x_1, 1) - m)) \} \}
\]

(Note that \( \hat{p}(m) \) is a given fixed sequence number.) Therefore, for each sequence number \( s \), \( A^s \) is an arithmetic subset of \( N^N \), too. Further, by the definition (4) we have

\[ A^{[a_0, a_1, \ldots, a_k]} = \bigcup_{n=0}^{\infty} A^{[a_0, a_1, \ldots, a_k, n]}, \]

where we denote \( \langle a_0 + 1, a_1 + 1, \ldots, a_k + 1 \rangle \) by \( [a_0, a_1, \ldots, a_k] \).

Now suppose that no member of \( A \) constitutes an arithmetic singleton. Let \( \alpha \in A \). Since \( A = \bigcup_{n=0}^{\infty} A^{[n]} \), there is an \( n_0 \) such that \( \alpha \in A^{[n_0]} \). \( A^{[n_0]} \) does not contain any arithmetic singleton, since its overset \( A \) does not. As seen above, \( A^{[n_0]} \) is also an arithmetic set and hence it contains no isolated elements. Therefore \( A^{[n_0]} \) is dense-in-itself. So, for each number \( m_0 \), \( A^{[n_0]} \cap \delta(\alpha([m_0])) \) is nonempty and dense-in-itself, where \( \delta(s) \) denotes the Baire interval determined by a sequence number \( s \). Let us put

\[ B^{[m_0]} = A^{[n_0]} \quad \text{and} \quad F_{[m_0]} = E_{[n_0]} \]

for all \( m_0 \). From each set \( B^{[m_0]} \cap \delta(\alpha([m_0])) \) we can choose an element \( \alpha_{[m_0]} \) such that the \( \alpha_{[m_0]} \)'s satisfy the following conditions:

\[ \alpha_{[m_0]} \neq \alpha, \quad \alpha_{[m_0]} \neq \alpha_{[m_0]'}, \quad \text{if} \quad m_0 \neq m_0'. \]
Since $B^{[m_0]} = \bigcup_{n=0}^{\infty} A^{[n_0,n_1]}$, for each $m_0$ there is an $n_1$ such that $\alpha_{[m_0]} \in A^{[n_0,n_1]}$. Let us put

$$B^{[m_0,m_1]} = A^{[n_0,n_1]} \quad \text{and} \quad F^{[m_0,m_1]} = E^{[n_0,n_1]}$$

for all $m_1$. Then $B^{[m_0,m_1]} \cap \delta(\tilde{a}_{[m_0]}([m_0+m_1+1]))$ is nonempty and dense-in-itself. From each set $B^{[m_0,m_1]} \cap \delta(\tilde{a}_{[m_0]}([m_0+m_1+1]))$, we can choose an $\alpha_{[m_0,m_1]}$ such that $\alpha_{[m_0,m_1]}$'s satisfy the following conditions:

$$\alpha_{[m_0,m_1]} \neq \alpha; \quad \alpha_{[m_0,m_1]} \neq \alpha_{[m_0,m_1]};$$
$$\alpha_{[m_0,m_1]} \neq \alpha_{[m_0,m_1]} \quad \text{if} \quad [m_0,m_1] \neq [m_0',m_1'].$$

And so on. Thus we obtain elements $\alpha_{[m_0,m_1,\ldots,m_k]} \in A$ for $k, m_i=0, 1, 2, \ldots$, and they possess the following properties:

$$(6) \quad \alpha_{[m_0,m_1,\ldots,m_k]} \neq \alpha_{[m_0,m_1,\ldots,m_k]} \quad \text{if} \quad [m_0, \ldots, m_k] \neq [m_0', \ldots, m_k'].$$
$$(7) \quad \alpha_{[m_0,m_1,\ldots,m_k]} \in \delta(\tilde{a}_{[m_0,m_1,\ldots,m_k]}([m_0+\ldots+m_k+1+k+1])).$$

Let $Q = \{\alpha_{[s]} | \text{Seq}(s) \text{ and } Lh(s)>0\}$. Then $Q$ is dense-in-itself and hence its derived set $Q'$ is a perfect set. Using (1)–(3) and (6)–(8) we can show that $Q'$ is contained in $A$. In proving this fact, note that each $E_s$ is a closed set. (For details, see Hahn [1, pp. 356–358].) Therefore $A$ contains a perfect subset. This completes the proof of Theorem 1.

Since the final expression for $\alpha_{\in A^{[m]}}$ in the preceding proof is also $\Pi_0^0$, we have shown that if $A$ is a nonempty $\Pi_0^0$ set with no $\Pi_0^0$ singleton then $A$ contains a perfect subset. Thus we obtain the following theorem:

**Theorem 3.** Every nonempty countable $\Sigma_0^{n+1}$ set contains a $\Pi_0^n$ singleton.

**References**


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1 This is based on a suggestion of Professor McLaughlin.