A GENERALIZATION OF MORI'S THEOREM

CHIN-PI LU

Abstact. In this article, we consider a generalization of Mori's theorem which is: Let $R$ be a Zariski ring; if the completion of $R$ is a unique factorization domain, then so is $R$.

Mori's theorem states that a Zariski ring $R$ is a unique factorization domain (UFD) if its completion $\hat{R}$ is a UFD. Validity of the theorem stems from the facts that $R$ is a Gelfand ring, a filtered ring whose radical is open, and that $\hat{R}$ is a faithfully flat $R$-module. From this point of view, a generalization of Mori's theorem is studied in this paper. We prove that if $R$ is a Gelfand ring whose completion $\hat{R}$ is a flat $R$-module and $R$ is a pure submodule of $\hat{R}$, in particular, $\hat{R}$ is a faithfully flat $R$-module, then $R$ is a UFD whenever $\hat{R}$ is a UFD. Applying the result, we also consider a generalization of Nagata-Mori's theorem [5].

In this paper, every ring is assumed to be a commutative ring with identity. A filtered ring $R$ with a filtration $\{q_n; n=0, 1, 2, 3, \cdots\}$ will be denoted by $(R, q_n)$, and its completion by $(\hat{R}, q_n)$, where $q_0=R$ and $q_n$ is the completion of $q_n$. We say that $R$ is hat-flat if $\hat{R}$ is a flat $R$-module (cf. [1]).

Definition 1. Let $E$ be a module over a ring $R$, and $F$ a submodule of $E$. $F$ will be called a pure submodule of $E$ if $rE\cap F=rF$ for all $r\in R$.

It is obvious that if the completion $\hat{R}$ of a filtered ring $R$ is a faithfully flat $R$-module, then $R$ is a pure submodule of $\hat{R}$.

Definition 2. A ring with a linear topology is called a Gelfand ring if its radical is open (cf. [3, p. 44]).

Note that a hat flat Gelfand ring is necessarily a separated topological ring by [3, Corollary (5.5)]. Evidently, a filtered ring $(R, q_n)$ is a Gelfand ring if and only if $q_1\subseteq \text{rad}(R)$. Thus every Zariski ring $R$ is a hat-flat Gelfand ring which is a pure submodule of $\hat{R}$ because $\hat{R}$ is a faithfully flat $R$-module.

The following lemma is easy to verify.

Lemma. Let $B$ be an ideal of a filtered ring $(R, q_n)$. Then $B$ is dense in $\hat{R}B$ for the $(\hat{q_n}B)$-topology.

Theorem 1. Let $(R, q_n)$ be a Gelfand ring which satisfies the following conditions: (i) $R$ is hat-flat and (ii) $R$ is a pure submodule of the $R$-module $\hat{R}$. Then $R$ is a UFD, if $\hat{R}$ is a UFD.
Proof. Firstly, $R$ satisfies the ascending chain condition for principal ideals because $\hat{R}$ does and $R$ is a pure submodule of $\hat{R}$. Next, let $a, b \in R$ and $P = Ra \cap Rb$. Since $R$ is hat flat, we have that $\hat{R}P = \hat{R}a \cap \hat{R}b$ by [2, p. 32, Proposition 6]; moreover, $\hat{R}P = \hat{R}c$ for some $c \in \hat{R}P$ as $\hat{R}$ is a UFD. According to the Lemma, there exists a $c_1 \in P$ such that $c = c_1 \mod \hat{q}_1P$, hence $\hat{R}c = \hat{R}c_1 + \hat{q}_1P$. Now put $\hat{R}c/\hat{R}c_1 = \hat{R}P/\hat{R}c_1 = E$, then $E$ is a finitely generated $\hat{R}$-module and $\hat{q}_1E = E$. By [3, Proposition (5.1)], $\hat{R}$ is a Gelfand ring, so that $\hat{q}_1 \subseteq \text{rad}(\hat{R})$. Applying Nakayama’s lemma we have $E = (0)$, that is, $\hat{R}c = \hat{R}c_1$. It follows that $P = Ra \cap Rb = Rc_1$, as $R$ is a pure submodule of $\hat{R}$. Thus the intersection of any two principal ideals of $R$ is principal, which implies that $R$ is a UFD.

Corollary 1. Let $R$ be a Gelfand ring such that $\hat{R}$ is a faithfully flat $R$-module. Then $R$ is a UFD, if $\hat{R}$ is a UFD.

The method of the proof of Theorem 1 is the same as the proof of Mori’s theorem in [5, p. 2]. We demonstrated that the method still works for the rings with conditions a bit weaker than Zariski rings. Simultaneously, we have shown that Theorem 1, as well as its Corollary 1, is a generalization of Mori’s theorem.

Corollary 2. Let $(R, q_n)$ be a Noetherian filtered ring satisfying either one of the following two conditions:

(i) $R$ is a hat-flat Gelfand ring,

(ii) the $(q_n)$-topology of $R$ is stronger than its radical topology and $\hat{R}$ is a Noetherian ring. Then $R$ is a UFD, if $\hat{R}$ is a UFD.

Proof. For case (i), the corollary follows from Corollary 1 to Theorem 1 and [3, Proposition (5.4)].

For case (ii), it is a result of Corollary 1 to Theorem 1 and [4, Proposition (5.4)].

Definition 3. A topological ring is said to be topologically artinian if it is equipped with a linear topology and there exists a fundamental system of neighborhoods of 0 consisting of ideals of finite length.

Proposition. A Gelfand ring $(R, q_n)$ which is topologically artinian is necessarily a quasi-semilocal ring.

Proof. Since $R$ is a Gelfand ring, $q_1 \subseteq \text{rad}(R)$. Moreover, $R/q_1$ is an artinian ring, hence there exist only a finite number of maximal ideals in $R/q_1$. Now, we can conclude that there exist only a finite number of maximal ideals in $R$, because every maximal ideal of $R$ contains $q_1 \subseteq \text{rad}(R)$.

Theorem 2. Let $R$ be a hat-flat Gelfand ring which is topologically artinian. Then $R$ is a UFD, if $\hat{R}$ is a UFD.
Proof. The radical of \( R \) is open as \( R \) is a Gelfand ring, whence every maximal ideal of \( R \) is open. Consequently, \( \tilde{R} \) is a faithfully flat \( R \)-module due to [1, Proposition 5]. Hence the proposition follows immediately from Corollary 1 to Theorem 1.

In the following Theorem 3 we consider a generalization of Nagata-Mori’s theorem [6, p. 56, Theorem 10].

**Theorem 3.** Let \((R, q_n)\) be a hat-flat ring which is either a Noetherian ring, or a separated topologically artinian ring, satisfying the ascending chain condition (a.c.c.) for principal ideals. If \( \tilde{R} \) is a UFD and \( S=1+q_n \) is generated by prime elements of \( R \), then \( R \) is also a UFD.

**Proof.** Suppose that \( R \) is a Noetherian hat-flat ring, and put \( R' = S^{-1}R \). Then \( R' \) is a Gelfand ring and \( \tilde{R} = \tilde{R}' \) by Theorem (5.1) and Corollary (5.2) both of [3]. Since \( \tilde{R} \) is a flat \( R \)-module, it is a flat \( R' \)-module by [3, Proposition (5.6)]. Thus \( R' \) is a Noetherian hat-flat Gelfand ring. Since \( \tilde{R} = \tilde{R}' \) is a UFD, \( R' \) is also a UFD by Corollary 2 to Theorem 1. Clearly, \( R \) is an integral domain which satisfies the a.c.c. for principal ideals, hence \( R \) is a UFD by Nagata’s theorem [5, Lemma 1.7]. Next, we assume the other case for \( R \). Clearly, \((R', S^{-1}q_n)\) is a Gelfand ring which is topologically artinian, and \( R \subseteq R' \subseteq \tilde{R} \) because \((\tilde{R}, q_n)\) is a Gelfand ring. This implies that \( \tilde{R}' = \tilde{R} \) since \((R', S^{-1}q_n)\) is a subspace of \((\tilde{R}, q_n)\) due to the fact that \( R \) is separated (cf. [6, p. 55, the proof of remark]). Moreover \( R' \) is hat-flat by [2, Proposition 6]. Thus \( R' \) is a hat-flat Gelfand ring which is topologically artinian. According to Theorem 2, \( R' \) is a UFD. Now, that \( R \) is a UFD also follows from Nagata’s theorem.\(^1\)

**Bibliography**


**Department of Mathematics, University of Colorado, Denver Center, Denver, Colorado 80202**

\(^1\)In Nagata’s theorem the ring need not be Noetherian if it satisfies the a.c.c. for principal ideals.