PRODUCTS OF UNCOUNTABLY MANY $k$-SPACES

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Abstract. It is shown that if a product of nonempty spaces is a $k$-space then for each infinite cardinal $n$ some product of all but $n$ of the factors has each $n$-fold subproduct $n$-$\mathcal{K}_0$-compact (each $n$-fold open cover has a finite subcover). An example is given, for each regular $\alpha$, of a space $X$ which is not $\alpha$-$\mathcal{K}_0$-compact (so $X^{\alpha+}$ is not a $k$-space) for which $X^\alpha$ is a $k$-space.

1. Introduction. A subset $F$ of a topological space $X$ is $k$-closed if $F \cap K$ is closed in $K$ for each compact subset $K$ of $X$. A space in which each $k$-closed subset is closed is called a $k$-space. (No separation axioms will be assumed, so this definition differs from some of the other published definitions.) Although conditions under which finite or countable products of $k$-spaces will be $k$-spaces have been extensively studied, for instance in [1], [2], [4], [6], and [7], the only noteworthy results concerning products of $k$-spaces having uncountably many factors are included in the fact, proved in [5], that for a product of nonempty $T_1$-spaces to be a $k$-space, some product of all but countably many of its factors must be countably compact. We improve and extend this result with:

Theorem. If a product of nonempty spaces is a $k$-space then, for each infinite cardinal $n$, some product of all but $n$ of its factors has each $n$-fold subproduct $n$-$\mathcal{K}_0$-compact.

Recall that a space is $n$-$\mathcal{K}_0$-compact if each $n$-fold open cover contains a finite subcover. As an immediate consequence of this theorem (together with Tychonoff’s Theorem) we have:

Corollary. All powers of a space $X$ are $k$-spaces if and only if $X$ is compact.

It is amusing to contrast this result with the fact, established in [8], that all powers of a $T_1$-space $X$ are normal if and only if $X$ is compact. (Thus all powers of a $T_1$-space $X$ are $k$-spaces if and only if all powers of

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X are normal.) The strength of our theorem on \( k \)-spaces is indicated by the following:

**Example.** For each regular cardinal \( \kappa \) there exists a space \( X \) such that \( X^\kappa \) is a \( k \)-space but \( X^m \) is not a \( k \)-space for any larger cardinal \( m \).

Indeed, \( X \) can be taken to be \( \kappa \). (As usual, a cardinal \( \kappa \) is identified with the smallest ordinal of cardinality \( \kappa \) and, unless otherwise indicated, is assumed to bear the order topology.) This space \( X \) is certainly not \( \kappa \)-\( \mathcal{K}_\omega \)-compact so, by the Theorem, \( X^m \) is not a \( k \)-space for any \( m \) greater than \( \kappa \). That \( X^\kappa \) is a \( k \)-space will follow from the more general considerations below.

Call a space \( \kappa \)-determined if a subset is closed whenever it meets each subset \( S \) having \( \kappa \) or fewer elements in a set which is closed in \( S \). Recall that a space is \( \kappa \)-bounded if each subset with \( \kappa \) or fewer elements is contained in a compact set. Clearly \( \kappa \)-boundedness is preserved by arbitrary products and each \( \kappa \)-bounded \( \kappa \)-determined space is a \( k \)-space.

**Proposition 1.** For \( \kappa \) an infinite cardinal, an \( m \)-fold product of \( \kappa \)-determined spaces is \( \kappa \)-determined if and only if all but at most \( \kappa \) of the factors are indiscrete.

We call a space \( \prec \kappa \)-bounded if each subset with fewer than \( \kappa \) elements is contained in a compact set and we call a space \( \prec \kappa \)-determined if a subset is closed whenever it meets each subset \( S \) having fewer than \( \kappa \) elements in a set which is closed in \( S \). Note that if \( \kappa = \kappa \) and \( \kappa \) is regular, then \( X \) is \( \prec \kappa \)-bounded and \( \prec \kappa \)-determined. Thus our next result shows that, for this \( X \), \( X^\kappa \) is a \( k \)-space.

**Proposition 2.** Let \( X = \prod \kappa \in \kappa^+ X_\kappa \). If each \( X_\kappa \) is \( \prec \kappa \)-bounded and \( \prec \kappa \)-determined, then \( X \) is a \( k \)-space.

2. **Proofs.**

**Proof of the Theorem.** The proof is by induction on \( \kappa \), so suppose that the Theorem holds for each cardinal less than \( \kappa \) and that \( X = \prod \kappa \in \kappa^+ X_\kappa \) is a nonempty \( k \)-space. In order to show that some product of all but at most \( \kappa \) of the factors of \( X \) has each \( \kappa \)-fold subproduct \( \kappa \)-\( \mathcal{K}_\omega \)-compact it suffices, by [5, Theorem 1], to show that all but \( \kappa \) of them must be \( \kappa \)-\( \mathcal{K}_\omega \)-compact. Suppose that this is not the case; since by the induction hypothesis all but at most \( \kappa \) of the factors are \( m \)-\( \mathcal{K}_\omega \)-compact for each \( \kappa \) less than \( \kappa \), we may suppose that each \( X_\kappa \) has an \( \kappa \)-fold open cover which has no subcover of smaller cardinality. Passing to complements of unions, each \( X_\kappa \) thus contains a nested family \( \{ A_\lambda : \lambda \in \kappa \} \) of nonempty closed sets with \( \bigcap \{ A_\lambda : \lambda \in \kappa \} = \emptyset \). Further, we may suppose that for each \( \alpha \) there exists a point \( y_\alpha \) in \( X \setminus A_\alpha \).

For each \( \lambda \) in \( \kappa \) let \( B_\lambda \) be the union, over all \( \gamma \) in \( \kappa \), of the product sets
whose \( \alpha \)th factor is \( \{y_\gamma\} \) for \( \gamma \leq \alpha \leq \gamma + \lambda \) and \( A_\lambda^\alpha \) otherwise. Let \( C_\lambda \) be the closure of \( \bigcup_{\beta \leq \lambda} B_\beta \) and set \( C = \bigcup_{\lambda \in \mathbb{n}} C_\lambda \); we will show that \( C \) is \( k \)-closed but not closed.

To see that \( C \) is not closed, note that since any finite subset of \( n^+ \) is contained in a segment \( [\gamma, \gamma + \lambda] \) for some \( \gamma \) and \( \lambda \), the point \( y = (y_\gamma) \) is in the closure of \( C \). On the other hand, \( y \) is not in \( C \) since, for \( \lambda \) in \( n \), \((X_0 \setminus A_0^\lambda) \times (X_{\lambda+1} \setminus A_{\lambda+1}^\lambda) \times \prod_{x \neq \{x \neq \lambda + 1} X_x \) is a neighborhood of \( y \) which does not meet \( \bigcup_{\beta \leq \lambda} B_\beta \), so \( y \) is not in the closure of \( C_\lambda \). Now let \( K = \prod_a X_a \) be compact, say \( K = \prod_a K_a \). We show that \( K \cap C \) is closed by showing \( K \cap C = K \cap C_\lambda \) for some \( \lambda \) — since \( C_\lambda \) is closed, this suffices. For each \( \alpha \), note that \( K_\lambda \) cannot meet cofinally many of the decreasing family \( \{A_\alpha^\lambda : \lambda \in n\} \) since its intersection is empty. Thus there exists a \( \lambda(\alpha) \) in \( n \) such that \( K_\lambda \cap A_\lambda^\alpha = \emptyset \) for each \( \lambda > \lambda(\alpha) \). Since the domain of \( \lambda \) is \( n^+ \) while its range is \( n \), there exists a \( \lambda_0 \) in \( n \) such that \( \{x : \lambda(x) = \lambda_0\} \) has cardinality \( n^+ \). For \( \lambda > \lambda_0 \), \( K \cap C_\lambda = K \cap C_{\lambda_0} \), since for each point \( x \) in the closure of \( \bigcup B_\beta : \beta < \gamma = \lambda \) \( x_\alpha \) is in \( A_{\alpha+1}^\lambda \) with fewer than \( n \) exceptions. Consequently \( K \cap C = K \cap C_{\lambda_0} \), so \( K \cap C \) is closed. This contradicts the hypothesis that \( \prod_a X_a \) is a \( k \)-space and thus completes the proof.

The proof above is a generalization of the proof sketched in [3, Exercise 7-J]. The first observation of our next proof implies that each subspace of an \( n \)-determined space is \( n \)-determined.

**Proof of Proposition 1.** Let us first note that if \( X \) is \( n \)-determined and \( x \) is in the closure of a subset \( A \) of \( X \), then \( x \) is in the closure of some \( n \)-fold or smaller subset of \( A \): Since an \( n \)-fold union of sets of cardinality \( n \) itself has cardinality \( n \), the operator which adjoins to \( A \) the closures of all of its \( n \)-fold subsets is idempotent, and is therefore the closure operator. Now suppose that \( X = \prod_{\alpha \in n} X_\alpha \) where each \( X_\alpha \) is \( n \)-determined and let \( x \) be in the closure of a subset \( A \) of \( X \). We will show that \( X \) is \( n \)-determined by showing that \( x \) is in the closure of some \( n \)-fold subset of \( A \).

Let \( F \) be any finite subset of \( n \). Since \( x \) is in the closure of \( A \), \( \pi_F(x) \) is in the closure of \( \pi_F(A) \), and hence, for some \( n \)-fold or smaller subset \( A_F \) of \( A \), \( \pi_F(x) \) is in the closure of \( \pi_F(A_F) \). Let \( A' = \bigcup \{A_F : F \subseteq n\} \) is finite and note that the cardinality of \( A' \) is less than or equal to \( n \). Since \( x \) is clearly in the closure of \( A' \), \( A' \) is as desired.

For the converse, suppose \( X = \prod_{\alpha \in n} X_\alpha \) where each \( X_\alpha \) contains a point \( x_\alpha \) and a closed subset \( F_\alpha \) with \( x_\alpha \) not in \( F_\alpha \). Let \( x \) be the point \( (x_\alpha) \) and let \( F \) be the set of points in \( X \) whose \( \alpha \)th coordinates, with at most \( n \) exceptions, lie in \( F_\alpha \). Clearly \( F \) meets each \( n \)-fold or smaller set in a closed set. Since \( x \) is in the closure of \( F \) but is not in \( F \), \( F \) is not closed, so this shows that \( X \) is not \( n \)-determined.

**Proof of Proposition 2.** Let \( A \subseteq X \) be \( k \)-closed and let \( x \) be any point in the closure of \( A \). We will produce a subset \( A' \) of such that \( x \) is in the
closure of $A'$ and such that, for each $\alpha$ in $\eta$, $\pi_\alpha A'$ has cardinality less than $\eta$. Since each $X_\alpha$ is $<\eta$-bounded, each $\pi_\alpha A'$, and hence $A'$ itself, is contained in a compact set. It follows that $x$ must be in $A$ and hence that $X$ is a $k$-space, as desired.

Let $\pi^\alpha$ denote the projection from $X$ to $X^\alpha = \prod_{\beta < \alpha} X_\beta$ and note that, since $\eta$ is regular, the proof of Proposition 1 adapts easily to show that $X^\alpha$ is $<\eta$-determined. We first show that, for each $\alpha$, $\pi^\alpha(x)$ is in $\pi^\alpha(A)$. Certainly $\pi^\alpha(x)$ is in the closure of $\pi^\alpha(A)$ and hence, since $X^\alpha$ is $<\eta$-determined, $\pi^\alpha(x)$ is in the closure of $\pi^\alpha(B)$ for some subset $B$ of $A$ having fewer than $\eta$ elements. Since $X$ is $<\eta$-bounded, $B$ is contained in some compact set $K$. Let $K_1$ be the projection of $K$ onto $\prod_{\beta \geq \alpha} X_\beta$, and let $K_2 = \pi^\alpha(K) \cup \{\pi^\alpha(x)\}$. Since $A$ is $k$-closed, $A \cap K_1 \times K_2$ is closed in $K_1 \times K_2$ and therefore its projection onto $K_2$, which is just $\pi^\alpha(A) \cap K_2$, is closed in $K_2$. Since $\pi^\alpha(B) \subseteq \pi^\alpha(A) \cap K_2$ and $\pi^\alpha(x)$ is in the intersection of the closure of $\pi^\alpha(B)$ with $K_2$, it follows that $\pi^\alpha(x)$ is in $\pi^\alpha(A)$, as desired.

To construct the set $A'$, choose, for each $\alpha$, a point $x^\alpha$ in $A$ such that $\pi^\alpha(x^\alpha) = \pi^\alpha(x)$ and let $A' = \{x^\alpha : \alpha \in \eta\}$. It is clear that $A'$ has the desired properties, so the proof is complete.

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