ON ALGEBRAS SATISFYING THE IDENTITY

\((yx)x + x(xy) = 2(xy)x\)

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Abstract. Simple, strictly power-associative algebras satisfying the identity \((yx)x + x(xy) = 2(xy)x\) over a field of characteristic not 2 or 3 have been classified by F. Kosier as commutative Jordan, quasi-associative, or of degree less than three. In the present paper those of degree three or greater are shown to be commutative, which eliminates the quasi-associative case mentioned above.

According to a result of F. Kosier [2, Theorem 4.7, p. 317], the simple, strictly power-associative algebras over a field of characteristic not 2 or 3 and satisfying the identity

\[(yx)x + x(xy) = 2(xy)x\]

may be characterized as being either of degree less than three, non-commutative Jordan, or quasi-associative. It will be shown in the following that this list of possibilities can be reduced and the following theorem is proved.

Theorem. A simple, strictly power-associative algebra over a field of characteristic not 2 or 3 and which satisfies (1) is

(a) a commutative Jordan algebra;
(b) an algebra of degree 2; or
(c) an algebra of degree 1.

To prove this theorem we will take advantage of the earlier mentioned result due to Kosier and assume in what follows that \(A\) is a simple, strictly power-associative algebra of degree exceeding 2 over a field of characteristic not two or three and satisfying (1). By that result, \(A\) is then either a Jordan algebra or a quasi-associative algebra and thus in either case is a non-commutative Jordan algebra [1, Theorem 2, p. 582]. Since the objective is to show that \(A\) is commutative and since \(A\) is commutative if and only if every scalar extension is commutative, we may assume that \(K\) is an algebraically closed field.

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Notations used here include \((x, y, z)\) to denote \((xy)z - x(yz)\) and \(x\cdot y\) to denote \(xy + yx\). Noncommutative Jordan algebras satisfy the identities

\begin{equation}
F(x, y, z) = 0 \quad \text{where} \quad F(x, y, z) = (x, y, z) + (z, y, x),
\end{equation}

\begin{equation}
J(x, y, z, w) = 0
\end{equation}

where

\[
J(x, y, z, w) = (x, y, z \cdot w) + (z, y, w \cdot x) + (w, y, x \cdot z).
\]

The identity (2) is the linearization of the flexible law, \((x, y, x) = 0\).

Advantage will be taken here of well-known ([1], [3]) facts regarding idempotents in a noncommutative Jordan algebra. Included in these is the vector space direct sum decomposition relative to any idempotent \(e\);

\[
A = A(e, 0) + A(e, 1) + A(e, 2)
\]

where \(A(e, 2) = \{x \in A : e \cdot x = \lambda x\}\) for \(\lambda = 0, 1, \text{or } 2\). Then \(A(e, 0) A(e, 2) = A(e, 2) A(e, 0) = 0\), the subspaces \(A(e, \lambda)\) are subalgebras for \(\lambda = 0\) or 2, and \(A(e, \lambda) A(e, 1) + A(e, 1) A(e, \lambda) \subseteq A(e, 1)\) for \(\lambda = 0\) or 2. The last property is referred to as stability. Also for \(\lambda = 0\) or 2 and \(x \in A(e, \lambda)\), \(2ex = 2xe = \lambda x\). Since the degree of \(A\) exceeds 2 and since \(K\) is algebraically closed, there are pairwise orthogonal idempotents \(e_1, e_2, \text{and } e_3\) such that \(e_1 + e_2 + e_3 = 1\). Relative to these three idempotents \(A\) has the decomposition, \(A = \sum A_{ij}, 1 \leq i \leq 3, j \leq 3\), where \(A_{ij} = A(e_i, 2)\) and for \(i \neq j\), \(A_{ij} = A_{ji} = A(e_i, 1) \cap A(e_j, 1)\). For \(i, j, k\) pairwise distinct, these subspaces have the properties \(A_{ii} A_{ij} + A_{ij} A_{ii} \subseteq A_{ij}, A_{ii} A_{jk} = A_{kk} A_{ij} = 0, A_{ij} A_{jk} \leq A_{ik}, A(e_i, 1) = A_{ij} + A_{ik}\), and \(A(e_i, 0) = A_{ij} + A_{jk} + A_{kk}\).

We shall adopt the notation that for \(e\) an idempotent, for \(\lambda = 0, 1, \text{and } 2\), and for \(S\) a subset of \(A\), \([S]_\lambda\) shall denote the set of all components in \(A(e, \lambda)\) of elements of \(S\). Similarly \([S]_{ij}\) denotes the set of components in \(A_{ij}\) of elements in \(S\). Then for subspaces \(S\) and \(T\), the commutative product \(S \circ T\) is defined as \(\sum_{i=0,1,2} [S + TS]_i\) and \(S^{(2)} = S \circ S\). Under this agreement, \(S \circ T\) contains \(ST \circ TS\) so that a subspace \(S\) is an ideal of \(A\) if \(A \circ S \subseteq S\).

If \(e\) is any idempotent then the subspace \(C(e)\) shall denote the set \(\{x \in A(e, 1) : 2ex = x\}\). Then \(M(e)\) denotes the subspace \(C(e) + C(e) \circ A(e, 1)\). The subspace \(C(e_i)\) is singled out for special attention and is denoted simply by \(C\). Similarly, \(M\) denotes \(M(e)\). The proof of the theorem stated above proceeds by showing that \(M\) is an ideal of \(A\). This fact along with the simplicity of \(A\) yields the equality \(C = A(e, 1)\). One can move then with reasonable dispatch to the commutativity of \(A\). It is necessary to first deduce some preliminary lemmas.

**Lemma 1.** If, for any idempotent \(e\), \(x\) and \(y\) are in \(A(e, 1)\) and \(z\) is in \(A(e, \lambda)\) for \(\lambda = 0\) or 2 then

\begin{equation}
[xy]_\lambda z = [x(yz) + (1 - \lambda)(ey - \frac{1}{4}\lambda y)(z \cdot x)]_\lambda
\end{equation}
and

\[(5) \quad z[x]_y = [(zy)x + (\lambda - 1)(ey - \frac{1}{2}\lambda y)(z \cdot x)]_x.\]

**Proof.** Expanding the identity \(J(x, y, e, z) - F(z, y, x) = 0\) yields \((xy)z = x(yz) + (1 - \lambda)(e, y, z \cdot x)\). Equating the components in \(A(e, \lambda)\) and noting that \([e(y(z \cdot x))]_x = \frac{1}{2}\lambda (y(z \cdot x))_x\) gives the identity (4). The identity (5) is obtained similarly by expanding \(J(x, y, e, z) - \lambda F(z, y, x) = 0\).

**Lemma 2.** If \(e\) is any idempotent then the subspace \(H(e) = A(e, 1) + A(e, 1)^{(2)}\) is an ideal of \(A\).

**Proof.** Stability and the definition of \(H(e)\) yield immediately that \(A \circ A(e, 1) \subseteq H(e)\) and that \(A(e, 1) \circ A(e, 1)^{(2)} \subseteq H(e)\). Let \(x, y, z\) be as in Lemma 1. Then \([xy]_x\) and \([xy]_z\) are in \(H(e)\) since the right members of the identities (4) and (5) are in \(H(e)\). Thus, for \(\lambda = 0\) or \(2\), \(A(e, \lambda) \circ A(e, 1)^{(2)} \subseteq H(e)\) and \(H(e)\) is an ideal.

**Lemma 3.** Relative to the idempotents \(e_1, e_2, \text{ and } e_3\) the equality \(A_{ij} = A_{ik} A_{kj} + A_{kj} A_{ik}\) holds.

**Proof.** By the previous lemma, \(H(e_i + e_j)\) is an ideal of \(A\). The simplicity of \(A\) yields \(H(e_i + e_j) = A\). The components, \(A_{ij}\) and \([H(e_i + e_j)]_{ij}\), of these spaces are then equal and

\[
[H(e_i + e_j)]_{ij} = [(A_{ik} + A_{jk})^2]_{ij} = A_{ik} A_{jk} + A_{jk} A_{ik}.
\]

**Lemma 4.** The subspace \([C(e_i)]_{ij}\) is contained in the subspace \(C(e_i)\).

**Proof.** If \(y\) is in \([C(e_i)]_{ij}\) then \(y + z = x\) for some \(z\) in \([C(e_i)]_{ik}\) and \(x\) in \(C(e_i)\). Then \(y + z = x = 2e_i x = 2e_i y + 2e_i z\), so since \(e_i y\) is in \(A_{ij}\) and \(e_i z\) is in \(A_{ik}\), it follows that \(y = 2e_i y\).

**Lemma 5.** If \(i \neq j\) then \([C(e_i)]_{ij} = [C(e_j)]_{ij}\).

**Proof.** Let \(x\) be in \([C(e_j)]_{ij}\). Then \(x e_i - e_i x = 2F(e_i, e_j, x) = 0\). This implies that \(x\) is in \([C(e_i)]_{ij}\). Since \(i\) and \(j\) are arbitrary this completes the proof.

From this point on, \(C_{ij}\) will denote the subspace \([C(e_i)]_{ij}\).

**Lemma 6.** If \(e\) is an idempotent, \(y\) in \(A(e, \lambda)\) for \(\lambda = 0\) or \(2\), and \(x\) in \(C(e)\) then \(xy = yx\).

**Proof.** Expanding \(2F(y, e, x) = 0\) yields \([1 - \lambda](xy - yx) = 0\) and since \(\lambda \neq 1\), \(xy = yx\).

**Lemma 7.** If \(y\) in \(A(e, \lambda)\) for \(\lambda = 0\) or \(2\) then \(yC(e) + C(e)y \subseteq C(e)\).
Proof. Let \( x \) be in \( C(e) \). By Lemma 6 and by identity (2), \( F(e, x, y) + (e - 1)(yx - xy) = 0 \). This expands to \( (yx)e = e(yx) \). Since \( yx \) is in \( A(e, 1) \) then \( yx \) is in \( C(e) \). That \( xy \) is in \( C(e) \) then follows from \( xy = yx \).

Lemma 8. The product \( A(e_1, 1)^{(2)} \) is contained in \( A(e_1, 0) + A(e_1, 2) \).

Proof. Since \( A(e_1, 1) = A_{12} + A_{13} \), the desired containment follows if the component in \( A_{ij} \) of \( (A_{ij})^2 \) is zero for \( j = 2 \) and 3. In (4) relative to the idempotent \( e_k \) where \( k \neq j \) and \( k = 2 \) or 3 let \( x \) and \( y \) be chosen, one each, from \( A_{1k} \) and \( A_{jk} \). Let \( z \) be in \( A_{ij} \) and let \( \lambda \) be 0. Then the right member of (4) has zero as its component in \( A_{ij} \) so \( [xy]_0z \) is in \( A_{11} + A_{ij} \). But since \( A_{ij} = A_{1k}A_{kj} + A_{kj}A_{1k} \) it follows that \( (A_{ij})^2 \subseteq A_{11} + A_{ij} \) and the desired result is achieved.

Lemma 9. The subspace \( M = C + A(e_1, 1) \cdot C \) is an ideal of \( A \).

Proof. It is immediate from Lemma 7 that \( A \cdot C \) is contained in \( M \). Since \( C \cdot A(e_1, 1) \subseteq A(e_1, 0) + A(e_1, 2) \) by Lemma 8 and since \( A(e_1, 0) \cdot A(e_1, 2) = 0 \) it suffices to show that \( A(e_1, 1) \cdot [C \cdot A(e_1, 1)] \) is contained in \( M \) for \( \lambda = 0 \) and 2 and to show that \( A(e_1, 1) \cdot [C \cdot A(e_1, 1)] \) is contained in \( M \). The first containment follows readily from the identities (4) and (5) since if \( z \) is in \( A(e_1, \lambda) \) and if \( x \) and \( y \) are selected in any order from \( C \) and \( A(e_1, 1) \) then the right members are in \( M \). The second containment may be obtained by considering the various subspaces \( A_{ij} \) and \( C_{ij} \) since \( A(e_1, 1) = A_{12} + A_{13} \) and \( C = C_{12} + C_{13} \). Since \( C_{ij} \subseteq C(e_k, 1) \) and \( A_{ijk} \subseteq A_{ik} \) for any \( i, j, \) and \( k \), \( A_{ij} \cdot C_{jk} \subseteq C_{ik} \) follows from \( A_{ij} \subseteq A(e_k, 0) \). Thus \( A_{ij} \cdot C_{jk} \subseteq C_{ik} \subseteq M \) for \( (j, k) \) equal to (2, 3) or (3, 2). By selecting \( z \) from \( A_{ij} \) and \( x \) and \( y \) one each from \( A_{ik} \) and \( C_{ik} \) in (4) and (5) relative to \( e_k \), it can be shown that \( A_{ij} \cdot (A_{ik} \cdot C_{ik}) \subseteq C_{ik} \subseteq M \) for \( (j, k) \) equal to (3, 2) or (2, 3). Finally, for \( (j, k) \) equal to (3, 2) or (2, 3), \( A_{ij} \cdot C_{kj} \) is in \( M \) by the proof being analogous to the argument earlier in the proof of this lemma that \( A_{ij} \cdot (A_{ik} \cdot C_{ik}) \) is in \( M \). Since \( A_{kj} \cdot C_{ij} \) is also contained in \( A(e_1, 1) \) it is in \( C_{kj} \). Therefore \( A_{ik} \cdot (A_{kj} \cdot C_{ij}) \subseteq A_{ik} \cdot C_{kj} \subseteq A_{ij} \subseteq M \). A similar argument yields \( A_{kj} \cdot (A_{ik} \cdot C_{ij}) \) in \( M \). This completes the proof of the lemma.

In a simple, power-associative, flexible algebra with orthogonal idempotents \( e \) and \( f \) the subspace \( A(e, 1) \) is not the zero subspace since otherwise \( A \) is the direct product of the ideals \( A(e, 0) \) and \( A(e, 2) \). For any element \( x \) in \( A(e, 1) \), \( ex = xe \) if and only if \( 2ex = x \). Let \( y \) be a nonzero element of \( A(e_1, 1) \). Then letting \( x \) in (1) be \( e_1 \) gives \( (ye_1)e_1 + e_1(e_1e_1) = 2(e_1y)e_1 \). By the flexibility of \( A \), \((e_1y)e_1 = e_1(ye_1)\) so \((ye_1)e_1 - (e_1y)e_1 = e_1(ye_1) - e_1(e_1y)\). Thus
(ye_1 - e_1 y)e_1 = e_1 (ye_1 - e_1 y) and hence depending on whether ye_1 - e_1 y is zero or not zero, either y or ye_1 - e_1 y is a nonzero element of C. Since \( M \cong C \) and \( A \) is simple, \( M = A \). This implies that \( A(e_1, 1) = C \).

**Lemma 10.** If \( x \) and \( y \) are in \( A(e_1, 1) \) then \( xy = yx \).

**Proof.** By the above, \( x \) and \( y \) are in \( C \). Expanding \( 2[F(e_1, x, y)]_\lambda = 0 \) for \( \lambda = 0 \) or \( 2 \) gives \( (1 - \lambda)[xy - yx]_\lambda = 0 \) so \( [xy]_\lambda = [yx]_\lambda \). Thus, since by Lemma 8, \( xy \) and \( yx \) are in \( A(e_1, 0)A + (e_1, 2) \), \( xy = [xy]_0 + [yx]_2 = [yx]_0 + [yx]_2 = yx \) proving the lemma.

By Lemmas 6 and 10, \( xy = yx \) for each \( x \) in \( A \) and \( y \) in \( A(e_1, 1) \). Thus to show that \( A \) is commutative it is only necessary to show that \( A(e_1, \lambda) \) is a commutative subalgebra for \( \lambda = 0 \) and \( 2 \). This is the substance of the final lemma.

**Lemma 11.** If \( x \) and \( y \) are in \( A(e_1, \lambda) \) for \( \lambda = 0 \) or \( 2 \) then \( xy = yx \).

**Proof.** By Lemma 2, \( H(e_1) \) is an ideal of \( A \) and by the simplicity of \( A \), \( H(e_1) = A \). Thus \( A(e_1, \lambda) \subseteq A(e_1, 1) \). Then the desired result follows if \( x(zw) = (zw)x \) for \( z \) and \( w \) in \( A(e_1, 1) \) and \( x \) in \( A(e_1, \lambda) \). But \( F(z, w, x) = 0 \) and by Lemmas 6 and 10, \( z(wx) = (xz)w \) so \( (zw)x = x(zw) = (zw)x - x(wz) = F(z, w, x) + z(wx) - (xz)w = 0 \). This shows that \( A(e_1, \lambda) \) is commutative.

Lemma 11 completes the argument that \( A \) is commutative and proves the above theorem.

**References**


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