ON SPANIER’S HIGHER ORDER OPERATIONS

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Abstract. Given a stack $A$ over a simplicial set $K$, we construct the generalized Eilenberg-Mac Lane complexes $K(A, n)$ that represents a stack cohomology theory over $K$. We show that the higher order operations defined by Spanier are some sections of $K(A, n)$ over $K$ for some stack $A$.

Introduction. Spanier in [2] defined higher order operations as obstructions to extending null-homotopies of maps carried by carriers over an abstract simplicial complex and showed that these operations are subsets of cohomology groups of the complex with coefficients in a “stack.” In this note we shall construct a stack cohomology theory over a simplicial set (semisimplicial complex) $K$ and prove that it is representable by generalized Eilenberg-Mac Lane complexes $K(A, n)$ constructed in §3. The higher order operations are then homotopy classes of cross sections of $K(A, n)$ over $K$. An example is given to show that when the stack $A$ is appropriately chosen, various cohomology operations can be represented by cross sections of $K(A, 1)$ over $I$, the standard 1-simplex $\Delta^1$.

1. Injective stacks. As in [1] a simplicial set $X$ is regarded as a category of simplexes with morphisms defined by face operators and degeneracy operators. For example, if $d^i x$ denotes the $i$th face of the $n$-simplex $x \in X_n$, then there is a morphism $d^i x : x \to d^i x$. A prestack over $X$ with values in a category $\mathcal{A}$ is by definition a contravariant functor $A : X \to \mathcal{A}$; $A$ is called a stack if, for every morphism $s^i x : x \to s^i x$ defined by the $i$th degeneracy operators $s^i$, $A(s^i x) : A(s^i x) \to A(x)$ is an isomorphism in $\mathcal{A}$. The category of prestacks over $X$ with values abelian groups is then a functor category; it is denoted by $s\mathcal{A}bX$.

Let $f : X \to Y$ be a simplicial map. Then $f$ is a (covariant) functor when the simplicial sets $X$ and $Y$ are regarded as categories. $f$ induces two functors $f^\#: \mathcal{A}bY \to \mathcal{A}bX$ and $f_\#: \mathcal{A}bX \to \mathcal{A}bY$, where $f^\#B = Bf$ and $(f_\#A)(y) = \prod_{x \in f^{-1}(y)} A(x)$, $x \in f^{-1}(y)$. $f^\#$ is (left) adjoint to $f_\#$ and both functors are exact. Consequently, $f_\#$ (resp. $f^\#$) preserves injectives (resp. projectives).

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Let $\Delta^n$ be the standard $n$-simplex (a simplicial set) with its only non-degenerate $n$-simplex denoted by $\delta^n$ and let $x: \Delta^n \to X$ be the simplicial map defined by the correspondence $\delta^n \to x, \quad x \in X_n$. Then the constant stack $Q_{(n)}$ over $\Delta^n$ with value the group of rationals mod 1 is an injective stack and therefore $x_{\#}Q_{(n)}$ is injective in $\mathcal{A}b_X$. Let $Q=\prod_{x \in X} x_{\#}Q_{(n)}$. Then $Q$ is an injective cogenerator of $\mathcal{A}b_X$ and therefore $\mathcal{A}b_X$ has enough injectives.

2. Stack cohomology. A prestack $A$ over $X$ is realized as a family of abelian groups $\{A(x); \ x \in X\}$ together with two families of homomorphisms

$$A(d^i): A(d^i x) \to A(x) \quad \text{and} \quad A(s^i): A(s^i x) \to A(x).$$

The cochain complex of $A$ is $\mathcal{C}A=\{C^nA=\prod_{x} A(x); \ x \in X_n\}$ with coboundary maps alternating sums of $A(d^i)$. The cohomology groups of $\mathcal{C}A$ are denoted $H^*(A)$ or $H^*(X; A)$. Every simplicial map $f:X \to Y$ induces homomorphisms $f^*:H^*(Y; B) \to H^*(X; f_{\#}B), \ B \in \mathcal{A}b_Y$, defined by $f^*([c])=[cf]$. Since the 0th cohomology functor $H^0$ is continuous and since $H^0(X; Q)=0$ for $q>0$ and $Q$ injective, $H^*$ are derived functors of $H^0$.

Let $K$ be a fixed simplicial set. Dual to the homology theory in [1] we have a cohomology theory over $K$ which can be computed by injective resolutions and which is unique in the sense of Eilenberg-Steenrod. For our use in this note we shall paraphrase only the definition and the homotopy axiom of the theory (in absolute case).

Let $\mathcal{C}_K$ be the category of simplicial sets over $K$, objects $X_\sigma$ are simplicial maps $\varphi:X \to K$; morphisms $f:X_\varphi \to Y_\psi$ are simplicial maps $f:X \to Y$ such that $\Psi f=\varphi$. Given a stack $A$ over $K$, the cohomology groups of $X_\varphi$ with coefficients in $A$ are $H^*(X_\varphi; A)=H^*(\varphi_{\#}A)$ as defined in the preceding paragraph. Thus for every map over $K$, $f:X_\varphi \to Y_\psi$, $f$ induces homomorphisms

$$f^*:H^*(Y_\psi; A) \to H^*(X_\varphi; A)$$

which are $f^*:H^*(\Psi_{\#}A) \to H^*(f_{\#}\Psi_{\#}A)=H^*(\varphi_{\#}A)$.

For each $q$-simplex $\sigma \in K_q$, let $\Delta^q_{\sigma}$ be the simplicial subset of $K$ generated by $\sigma$ and let $i_{\sigma}$ be the inclusion map. A homotopy over $K$ is a map $F:\ (X \times I)_{\#}\psi \to Y_{\#}\psi$ in $\mathcal{C}_K$, where $p(x, \delta)\equiv x$. It is a family of simplicial homotopies $F=\{F_{\sigma}:(i_{\sigma}^#X) \times I \to i_{\sigma}^#Y; \ \sigma \in K\}$.

3. The representation theorem. For $\tau=d\sigma$ a face of $\sigma \in K_q$, there is a map $j: \Delta^q_{\tau^{-1}} \to \Delta^q_\sigma$ in $\mathcal{C}_K$ defined by $j(\delta^{-1})=d\delta^{\sigma}$ as in the commutative diagram

$$\begin{array}{ccc}
\Delta^{q-1} & \xrightarrow{j} & \Delta^q \\
\downarrow \tau & & \downarrow \sigma \\
K & & K
\end{array}$$
$j$ induces a homomorphism of groups of normalized $n$-cocycles

$$Z^n(j): Z^n(\Delta^q; \sigma^#A) \to Z^n(\Delta^{q-1}; j^#\sigma^#A) = Z^n(\Delta^{q-1}; \tau^#A).$$

Similarly, for a degeneracy operator $s$ and $\sigma = sr$, one defines a homomorphism $Z^n(h)$. As a generalization of Eilenberg-Mac Lane complexes, we define simplicial sets $K(A, n) = \{K_q(A, n); q = 0, 1, 2, \ldots \}$ with $K_q(A, n) = \bigcup_\sigma Z^n(\Delta^q; \sigma^#A)$, $\sigma \in K_q$, and face operators and degeneracy operators defined by $Z^n(j)$ and $Z^n(h)$ respectively. An Eilenberg-Mac Lane object $K(A, n)_\theta$ is then a simplicial map $\theta: K(A, n) \to K$ that sends all cocycles in $Z^n(\Delta^q; \sigma^#A)$ onto $\sigma$. In particular, if $K = \Delta^0$ is a point and if $A$ is the constant stack over $K$ with value group $\pi$, then $K(A, n)$ is the Eilenberg-Mac Lane complex $K(\pi, n)$. Notice also that $K(A, n)_\theta$ can be viewed as a functor (covariant!) $E: K \to \text{Set}$ with $E(\sigma) = Z^n(\Delta^q; \sigma^#A)$.

An element $[c]$ in $H^n(K(A, n)_\theta; A) = H^n(\theta^#A)$ is said to be characteristic for $K(A, n)$ if it is represented by the fundamental cocycle $c$ defined by $c(e) = e(\delta^n)$ for every $e \in K_n(A, n)$. Let $[X_\varphi, K(A, n)_\theta]$ be the group of homotopy classes of maps over $K$ from $X$ to $K(A, n)$. Then

**Theorem A.** $[X_\varphi, K(A, n)_\theta]$ is naturally isomorphic to $H^n(X; K(A, n)_\theta)$.

Indeed, $[f] \mapsto f^*([c])$, where $c$ is a fundamental cocycle, defines an isomorphism $c_n: [X_\varphi, K(A, n)_\theta] \to H^n(X; K(A, n)_\theta)$ with $c_n^{-1}([h]) = [f]$ in $[X_\varphi, K(A, n)_\theta]$ defined by the equality $(f(x))(\delta^n) = h(x)$.

In particular, when $K = \Delta^0$ is a point, our results show that the functors $[\_ , K(A, n)_\theta]$ are derived functors of $H^0(\_; K(A, n)_\theta)$.

4. Higher order operations. In this section we shall paraphrase the main result and some examples in [2] in terms of cross sections of $K(A, n)_\theta$ over $K$.

Let $\mathcal{A}$ be the category of pointed topological spaces and base point preserving continuous maps and let $T: K \to \mathcal{A}, T(\sigma) = (\sigma)$, be the stack of geometrical realizations. For a carrier $\Phi: K \to \mathcal{A}$ (a stack!), a stack map $f: T \to \Phi$ is said to be carried by $\Phi$. A homotopy $F: f \simeq g$ of the maps $f$ and $g$ carried by $\Phi$ is a stack map

$$F = \{F_\sigma: T(\sigma)xI \to \Phi(\sigma); F_\sigma(x, 0) = f(x), F_\sigma(x, 1) = g(x)\}.$$

Let $\Gamma_n$ be the stack of groups defined by $\Gamma_n(\sigma) = \pi_n(\Phi(\sigma))$. Then the $(n+1)$-operation $O^n(f)$ is the set of obstructions $c_n(F)$ to extending null-homotopies $F$ on the $(n-1)$-skeleton $K^{n-1}$ of $K$ and so is a subset of $H^n(K; \Gamma_n)$. Let $X_\varphi$ in Theorem A be $1: K \to K$. Then

$$H^n(K; \Gamma_n) \approx [K, K(\Gamma_n, n)_\theta] \quad \text{for } n \geq 2.$$
and

**Theorem B.** The \((n+1)\)-operation \(O^n(f)\) (if it exists) is a subset of homotopy classes of cross sections of \(K(\Gamma_n, n)\).

When \(n=1\), various cohomology operations can be represented by cross sections of \(K(A, 1)\) over \(I\) when \(A\) is appropriately chosen. For example, a cross section representation of the Massey triple product is obtained by defining \(A\):

\[
H^{p+q-1}(X; G_{12}) \xrightarrow{d} H^{p+q+r-1}(X; G_{12}) \xleftarrow{d'} H^{p+q+r-1}(X; G_{23})
\]

with \(d(u) = u \cup u_3\), \(d'(v) = u_1 \cup v\), \(u_1 \cup u_2 = 0 = u_2 \cup u_3\), and the \(G\)'s suitably paired.

**References**


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