SEPARATING $p$-BASES AND TRANSCENDENTAL EXTENSION FIELDS

J. N. MORDESON AND B. VINOGRADE

Abstract. Let $L/K$ denote an extension field of characteristic $p \neq 0$. It is known that if $L/K$ has a finite separating transcendence base, then every relative $p$-base of $L/K$ is a separating transcendence base of $L/K$. In this paper we show that when every relative $p$-base of $L/K$ is a separating transcendence base of $L/K$, then the transcendence degree of $L/K$ is finite. We also illustrate the connection between the finiteness of transcendence degree of $L/K$ and the property that $L/K(X)$ is separable algebraic for every relative $p$-base $X$ of $L/K$.

Let $L/K$ denote an extension field of characteristic $p > 0$. If $X$ is a relative $p$-base such that $L/K(X)$ is separable algebraic, then we call $X$ a separating relative $p$-base. When every relative $p$-base of $L/K$ is a separating relative $p$-base we say that $L/K$ is of type $R_s$. Let $S$ denote the set of all intermediate fields of $L/K$. When every element of $S$ is of type $R_s$ (with respect to $K$), we say that $L/K$ is of type $R_s(S)$. This notation extends that used by the authors in [4] where a purely inseparable extension $L/K$ is called type $R$ when $L=K(X)$ for every relative $p$-base $X$, and where it is shown that $L/K$ is of type $R(S)$ if and only if $L/K$ has an exponent.

In this paper we give four theorems that illustrate the connection between type $R_s$ and the finiteness of transcendence degree. We make use of relevant results that appear in Mac Lane [3] and Dieudonné [1].

Finitely generated extensions, whose measures of inseparability have recently been analyzed anew by Kraft [2], are a subset of extensions of type $R_s(S)$, a fact that follows easily from Theorem 2 below.

Lemma. $L/K$ is of type $R_s$ if and only if there is no intermediate field $L'$ of $L/K$ such that $L=L'/(L)^{p}$ and $L/L'$ is not separable algebraic.

Proof. Suppose $L/K$ is not of type $R_s$. Then there exists a relative $p$-base $X$ which is not a separating relative $p$-base. Hence if we set $L'=K(X)$, then $L=L'(L)^{p}$ and $L/L'$ is not separable algebraic. On the other hand, suppose there exists an intermediate field $L'$ of $L/K$ such that

Received by the editors April 5, 1971.

AMS 1970 subject classifications. Primary 12F20; Secondary 12F99.

Key words and phrases. Extension fields, separating transcendence bases, relative $p$-bases.

© American Mathematical Society 1972
$L = L'(L^p)$ and $L/L'$ is not separable algebraic. Then $L'$ contains a relative $p$-base of $L/K$ which is not a separating relative $p$-base. Q.E.D.

**Corollary.** If $L/K$ is of type $R_s$, then $L/L'$ is of type $R_s$ for every intermediate field $L'$ of $L/K$.

**Proof.** Suppose $L/K$ is of type $R_s$ and that $L/L'$ is not of type $R_s$ for some intermediate field $L'$ of $L/K$. Then there exists an intermediate field $L''$ of $L/L'$ such that $L = L''(L^p)$ and $L/L''$ is not separable algebraic. This is a contradiction because $L''$ is also an intermediate field of $L/K$. Q.E.D.

We call $L/K$ separable when the tensor product $L \otimes_R K^{p^{-1}}$ is a field.

**Theorem 1.** When $L/K$ is separable, the following statements are equivalent.

1. $L/K$ is of type $R_s(S)$.
2. $L/K$ is of type $R_s$.
3. $L/K$ has a finite separating transcendence base.
4. Every relative $p$-base of $L/K$ is a separating transcendence base of $L/K$.
5. Every relative $p$-base of $L/K$ is a transcendence base of $L/K$.
6. The transcendence degree of $L/K$ equals the imperfection degree of $L/K$, and these are finite.

**Proof.** (1)$\Rightarrow$(2)$\Rightarrow$(3). That (1) implies (2) is immediate. Suppose (2) holds. Then, by [3, Theorem 11, p. 381], $L/K$ has a separating transcendence base $T$. Suppose $T$ is infinite. Let $T_0 = \{t_1, t_2, \cdots \}$ be a denumerable subset of $T$ and set $T' = T - T_0$ (set difference), $K' = K(T')$. Then $T_0$ is a separating transcendence base of $L/K'$. $T_0$ is therefore a relative $p$-base of $L/K'$, hence the set $T_0' = \{t_1t_2^p, t_2t_3^p, \cdots \}$ is a relative $p$-base of $L/K'$. By our corollary, $L/K'(T_0')$ is separable algebraic. Hence $t_1 \in K'(T_0', t_1^p)$. However this contradicts the algebraic independence of $T_0$ over $K'$. Thus $T$ is finite.

(3)$\Rightarrow$(4). This implication follows from [3, Corollary, p. 385].

(4)$\Rightarrow$(5). This follows from [3, Theorem 13, p. 383].

(5)$\Rightarrow$(6). That the transcendence degree of $L/K$ equals the imperfection degree of $L/K$ is immediate. The finiteness condition follows from the equivalence of (4) and (5) and the proof of (2) implies (3).

(6)$\Rightarrow$(1). By [3, Theorem 11, p. 381], we have (3). Hence an application of [3, Theorem 17, p. 386] and [3, Corollary, p. 385] yields (1). Q.E.D.

When there exists an integer $e \geq 0$ such that $K(L^p)^e/K$ is separable but $K(L^{p^{e+1}})/K$ is not, then $e$ is called the inseparability exponent of $L/K$, as in [2, p. 111]. When $L/K$ has an inseparability exponent, there exist certain maximal separable intermediate fields of $L/K$ whose construction (for our
case) is indicated by Dieudonné [1, p. 17] (see also [3, p. 384]) as follows: From a relative $p$-base $X$ of $L/K$ select a subset $Y$ such that $Y^p$ is a relative $p$-base of $K(L^p)/K$. Since the latter extension is separable, $Y$ is algebraically independent over $K$, and since $K(L^p)/K(Y^p)$ is separable, so is $K(L^p, Y)/K(Y)$. Set $F = K(L^p, Y)$. Then $F/K$ is a maximal separable intermediate field of $L/K$ and $L/K$ is isomorphic over $F$ to a subfield of the field $F(\bigcap_{i} K^{p^{-\omega}})$. Such an intermediate field Dieudonné has called distinguished maximal separable. When $L/K$ has a finite relative $p$-base, the degree of $L$ over any distinguished maximal separable intermediate field is Weil's order of inseparability of $L/K$ ([1, pp. 14, 17], [2, p. 111]). The use of the term "distinguished" is consistent with that used by the authors in [5]. This follows from application of [5, Proposition 1.10, p. 5] to the fact that $Y$ is a relative $p$-base of $F/K$ and relatively $p$-independent in $L/K$.

In the finitely generated case, a distinguished maximal separable intermediate field $F$ is the same as the optimal separable intermediate field denoted by $K_0$ in [2, p. 111]. In fact, $K(F^p) = K(L^p)$ holds in our more general context.

For a transcendence base $T$ of $L/K$, let $S_T$ denote the maximal separable intermediate field of $L/K(T)$.

**Theorem 2.** When $L/K$ is arbitrary, the following statements are equivalent.

1. $L/K$ is of type $R_s(S)$.

2. $L/K$ has finite transcendence degree and $L/S_T$ has an exponent for every transcendence base $T$ of $L/K$.

3. $L/K$ has finite transcendence degree and $L/S_T$ has an exponent for some transcendence base $T$ of $L/K$.

4. $L/K$ has an inseparability exponent $e$ and $K(L^p)/K$ has a finite separating transcendence base.

5. $L/K$ has a distinguished maximal separable intermediate field of type $R_s$.

6. $L/K$ has a distinguished maximal separable intermediate field and every such field is of type $R_s$.

**Proof.** (1)$\Rightarrow$(2)$\Rightarrow$(3). If $T$ is any transcendence base of $L/K$, then (1) implies that $L'/K$ is of type $R_s$ for every intermediate field $L'$ of the purely inseparable extension $L/S_T$. Since by our corollary $L'/S_T$ is also of type $R_s$, $L'/S_T$ is actually of type $R$. Thus $L/S_T$ is of type $R(S)$, hence $L/S_T$ has an exponent by [4, Corollary, p. 240]. Since $S_T/K$ is also of type $R_s$, $T$ is finite by Theorem 1 above. Thus (1) implies (2). That (2) implies (3) is trivial.

(3)$\Rightarrow$(1). If $L'$ is an intermediate field of $L/K$, then a transcendence base $Z'$ of $L'/K$ can be extended to a transcendence base $Z$ of $L/K$. Let $T$

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use
be the transcendence base of \( L/K \) satisfying (3). Since \( T \) is finite, there exists a positive integer \( m \) such that \( T^{p^m} \subseteq S_Z \). Hence \( L/S_Z \) has an exponent. Now \( S_Z \supseteq S_{Z'} \), \( S_Z / K(Z') \) is separable and \( S_{Z'} / K(Z') \) is, in particular, relatively perfect. Hence \( S_Z / S_{Z'} \) is separable by [1, Proposition 6, p. 8]. Thus \( L \cap S_Z = S_{Z'} \). Since \( L/S_Z \) has an exponent, say \( n \), \( L^{p^n} \subseteq L \cap S_Z = S_{Z'} \). Thus property (3) is inherited by every intermediate field of \( L/K \). Hence it suffices to show that (3) implies \( L/K \) is of type \( R_s \). Now \( T \) is finite, so \( S_T/K \) is of type \( R_s(S) \) by Theorem 1. Since \( L^{p^n} \subseteq S_T \) for some integer \( e \geq 0 \), we have that \( K(L^{p^n})/K \) is of type \( R_s \). If \( X \) is any relative \( p \)-base of \( L/K \), \( X^{p^e} \) contains a relative \( p \)-base of \( K(L^{p^n})/K \). Hence \( K(L^{p^n})/K(X^{p^e}) \) is separable algebraic, whence \( K(L^{p^n}, X)/K(X) \) is separable algebraic. Since \( L \supseteq K(L^{p^n}, X) \) and \( X \) was arbitrary, we have that \( L/K \) is of type \( R_s \).

(3) \( \iff \) (4). That (4) implies (3) follows easily. To show that (3) implies (4), note that by (3), \( L/K \) has an inseparability exponent, say \( e \). Since (3) and (1) are equivalent, \( K(L^{p^n})/K \) is of type \( R_s \). Hence, by Theorem 1, \( K(L^{p^n})/K \) has a finite separating transcendence base.

(4) \( \iff \) (5). To show (5) implies (4), let \( F = K(L^{p^n}, Y) \) be a distinguished maximal separable intermediate field such that \( F/K \) is of type \( R_s \). Since \( F/K \) is separable, every relative \( p \)-base of \( F/K \) is a finite separating transcendence base. Hence \( Y \) is a finite separating transcendence base of \( F/K \). Thus \( Y^{p^n} \) is a finite separating transcendence base of \( K(L^{p^n})/K \). To show that (4) implies (5), note that \( L/K(L^{p^n}) \) has an exponent and \( K(L^{p^n})/K \) has a finite separating transcendence base. By (3) implies (1), \( L/K \) is of type \( R_s(S) \). Hence \( F/K \) is of type \( R_s \).

(4) \( \iff \) (6). (6) implies (5) trivially and we have proved (5) implies (4). Hence (6) implies (4). Assume (4). Now, all distinguished maximal separable intermediate fields contain \( K(L^{p^n})/K \). Hence the proof that (4) implies (5) applies to every such distinguished intermediate field. Q.E.D.

**Theorem 3.** When \( L/K \) has finite transcendence degree, the following statements are equivalent.

1. \( L/K \) is of type \( R_s \).
2. \( L/S_T \) is of type \( R \) for every transcendence base \( T \) of \( L/K \).
3. \( L/S_T \) is of type \( R \) for some transcendence base \( T \) of \( L/K \).

**Proof.** That (1) implies (2) follows from the Corollary. That (2) implies (3) is trivial. To show that (3) implies (1), let \( X \) be a relative \( p \)-base of \( L/K \). Then \( X \) contains a relative \( p \)-base of \( L/S_T \). Since \( L/S_T \) is of type \( R \), \( L=S_T(X) \supseteq K(T, X) \). Thus \( L/K(T, X) \) is separable algebraic and by hypothesis \( T \) is finite. If \( L/K(X) \) is not separable algebraic, then \( K(T, X)/K(X) \) is not separable algebraic. Then \( K(T, X)/K(X) \) has a non-empty relative \( p \)-base, because \( K(T, X)/K(X) \) is finitely generated. But
since $L/K(T, X)$ is separable algebraic, $L/K(X)$ has a nonempty relative $p$-base, contrary to the fact that $L/K(X)$ is relatively perfect. Q.E.D.

When an extension field has a separating transcendence base, we say it is separably generated.

**Theorem 4.** When $L/K$ contains a separable intermediate field $F|K$ such that $L/F$ is finite degree purely inseparable, the following statements are equivalent.

1. $L/K$ is of type $R_s(S)$.
2. $L/K$ is of type $R_s$.
3. The transcendence degree of $L/K$ is finite and $F|K$ is separably generated.
4. The imperfection degree of $L/K$ is finite and $F|K$ is separably generated.
5. $F|K$ is of type $R_s$.
6. $K(L^{p^e})|K$ is of type $R_s$ for some integer $e \geq 0$.

**Proof.** (1) $\Rightarrow$ (2) $\Rightarrow$ (3). That (1) implies (2) is immediate. Suppose (2) holds. Let $X$ be a relative $p$-base of $L/F$. Then $X^{p^e} \subseteq F$ for some integer $e \geq 0$ and $X$ is finite. Let $Y$ be a relative $p$-base of $F/K$. Since $L/K$ is of type $R_s$ and $X \cup Y$ contains a relative $p$-base of $L/K$, $L/K(X, Y)$ is separable algebraic. Hence $K(L^{p^e}, Y)/K(X^{p^e}, Y)$ is separable algebraic. Now $K(L^{p^e}, Y) \subseteq F = K(F^{p^e}, Y) \subseteq K(L^{p^e}, Y)$. Thus $F = K(L^{p^e}, Y)$, so $F/K(X^{p^e}, Y)$ is separable algebraic. Since $F/K$ is separable, $F/K(Y)$ is separable. Hence $K(Y, X^{p^e})/K(Y)$ is separable and finitely generated. If the latter extension has a nonempty relative $p$-base, then we contradict the fact that $F/K(Y)$ is relatively perfect and $F/K(Y)$ is separably generated. Hence $K(Y, X^{p^e})/K(Y)$ is separable algebraic, so $F/K(Y)$ is of type $R_s$. Since $F/K$ is also separable, it has a finite separating transcendence base by Theorem 1. Since $L/F$ is algebraic, $L/K$ has a finite transcendence base.

(3) $\Rightarrow$ (4) $\Rightarrow$ (5). That (3) implies (4) is routine. Suppose (4) holds. Then the transcendence degree of $F/K$ must be finite. Hence by Theorem 1, $F/K$ is of type $R_s$.

(5) $\Rightarrow$ (6). Let $e$ be a positive integer such that $L^{p^e} \subseteq F$. Since $F/K$ is of type $R_s$, $F/K$ is of type $R_s(S)$. Hence $K(L^{p^e})/K$ is of type $R_s$.

(6) $\Rightarrow$ (1). Since $K(L^{p^e})/K$ is of type $R_s$, $K(L^{p^e})/K(X^{p^e})$ is separable algebraic for every relative $p$-base $X$ of $L/K$. Hence $K(L^{p^e}, X)/K(X)$ is separable algebraic, that is $L/K$ is of type $R_s$. Hence, as in the proof of (2) implies (3), $L/K$ has finite transcendence degree and $F/K$ is of type $R_s$. Thus, replacing $S_T$ by $F$ in Theorem 2, we have that (1) holds. Q.E.D.

**References**


**Department of Mathematics, Creighton University, Omaha, Nebraska 68131**

**Department of Mathematics, Iowa State University, Ames, Iowa 50010**