A REPRESENTATION THEOREM FOR \( L^p \) SPACES

MAREK KANTER

Abstract. Using the theory of symmetric stable process of index \( p \in (0, 2]\), we prove that if a separable Fréchet space \( L \) has all its finite dimensional subspaces linearly isometric with a subspace of \( L^p[0, 1] \) then \( L \) itself is linearly isometric with a subspace of \( L^p[0, 1] \).

1. Summary. Let \( p \) be a real number \( \geq 1 \). Bretagnolle et al. [1] have proved that if \( L \) is a separable Banach space such that every finite dimensional subspace of \( L \) is linearly isometric with a subspace of \( L^p[0, 1] \) then in fact \( L \) is itself linearly isometric with a subspace of \( L^p[0, 1] \). We propose to give a simpler, more constructive proof of this fact which is in fact valid for \( p \in (0, 2] \). Our techniques are an extension of the techniques in Bretagnolle et al. [1].

2. Main results. If \( \Omega \) is any set and \( \mathcal{A} \) is a \( \sigma \)-field of subsets of \( \Omega \), then for any nonnegative measure \( \mu \) on \( (\Omega, \mathcal{A}) \) we denote, for any \( p > 0 \), \( L^p[\Omega, \mu] \) to be the set of all real valued \( \mathcal{A} \) measurable functions \( f \) on \( \Omega \) such that \( \int_{\Omega} |f(s)|^p \, d\mu(s) < \infty \). We identify functions equal a.e. For \( p > 0 \) we define the norm of \( f \), denoted by \( \|f\| \), to be \( \left( \int_{\Omega} |f(s)|^p \, d\mu(s) \right)^{1/p} \). For \( p \geq 1 \), the norm \( \|f\| \) makes \( L^p[\Omega, \mu] \) into a Banach space, while for \( p \leq 1 \) the metric \( \|f\|_p \) makes \( L^p[\Omega, \mu] \) into a Fréchet space. If \( \Omega = [0, 1] \) and \( \mu \) is Lebesgue measure we simply write \( L^p[0, 1] \). In the following theorem, when referring to isometries we mean the metric \( \|f\| \) when \( p \geq 1 \) and \( \|f\|_p \) when \( p < 1 \). If \( E \) is a topological space and if \((\mu, \mu_1, \mu_2, \ldots)\) is a sequence of bounded measures on \( E \), we say that \( \mu_n \) converges weakly to \( \mu \) if \( \int_E f \, d\mu_n \to \int_E f \, d\mu \) for all bounded continuous real functions on \( E \).

**Theorem.** Suppose \( L \) is a separable Fréchet space such that any finite dimensional subspace of \( L \) is linearly isometric with a subspace of \( L^p[0, 1] \), where \( p \) is a fixed real number in \((0, 2] \). Then in fact \( L \) is itself linearly isometric with a subspace of \( L^p[0, 1] \). (If \( \rho \) denotes the metric on \( L \) and \( H \) is a linear isometry from a subspace \( L_0 \) of \( L \) to \( L^p[0, 1] \), then for \( p \geq 1 \) we have \( \rho(f) = \|H(f)\| \) for \( f \in L_0^1 \), while for \( p < 1 \) we have \( \rho(f) = \|H(f)\|_p \).

Received by the editors April 5, 1971.

AMS 1970 subject classifications. Primary 42A36, 46E30, 60B05, 60B10.

Key words and phrases. Linear isometry, symmetric stable process, weak convergence of measures.

© American Mathematical Society 1972

472
$H$ of course depends on the subspace $L_0$ but $\|H(f)\|$ does not, so we suppress this dependence from our notation.\)

**Proof.** For any integer $n>0$ and any choice of $f_1, \cdots, f_n \in L$, the function $\exp(-\|t_1 H(f_1) + \cdots + t_n H(f_n)\|^p)$ defined for $t=(t_1, \cdots, t_n) \in \mathbb{R}^n$ is in fact the characteristic function of a probability distribution in $\mathbb{R}^n$ which is symmetric stable of index $p$. (This is most easily seen by letting $X=(X(v))_{v \in [0, 1]}$ be a stochastic process such that $X(0)=0$ and such that $X$ has time homogeneous independent increments with

$$E(\exp(i\alpha X(v))) = \exp(-|\alpha|^p),$$

and then defining the stochastic integrals $X_m = \int_{[0, 1]} H(f_m) \, dX$, for $1 \leq m \leq n$, as in M. Schilder [4]. We then have

$$E(\exp(i(t_1 X_1 + \cdots + t_n X_n))) = \exp(-\|t_1 H(f_1) + \cdots + t_n H(f_n)\|^p).$$

By [3] or [1] it follows that there exists a Borel measure $\mu_n$ on the unit sphere $S_n$ of $\mathbb{R}^n$ such that

$$\|t_1 H(f_1) + \cdots + t_n H(f_n)\|^p = \int_{S_n} |\langle t, s \rangle|^p \, d\mu_n(s)$$

where for $s=(s_1, \cdots, s_n) \in S_n$ we let $\langle t, s \rangle = t_1 s_1 + \cdots + t_n s_n$. Now $1 = \sum_1^n (s_m)^2$ hence $1 \leq \sum_1^n |s_m|^p$ since $p \leq 2$. It follows that

$$\mu_n(S_n) \leq \int_{S_n} \sum_1^n |s_m|^p \, d\mu_n(s) = \sum_1^n \|H(f_m)\|^p.$$

Let us now choose a sequence $\{f_m\}_{m=1}^\infty$ in $L$ such that the closure of the linear span of $\{f_m\}$ is all of $L$ and such that $\sum_1^\infty \|H(f_m)\|^p < \infty$. Let $E$ denote the set of all sequences $s=(s_1, \cdots, s_n, \cdots)$ of real numbers $s_n$ such that $\sum_1^\infty (s_m)^2 \leq 1$. Let $E$ have the smallest topology which makes all the functionals $s \mapsto \langle t, s \rangle$ continuous where $t$ varies over the set of all finite sequences $(t_1, \cdots, t_n)$ and $(t, s) = \sum_1^n t_m s_m$ for $s \in E$; let $\mathcal{B}_E$ stand for the $\sigma$-field of subsets of $E$ generated by the open subsets of $E$. Now it is well known that $E$ is compact [2, p. 427], and it is clear that $S_n$ is a closed subset of $E$. The measure $\mu_n$ defined before is now to be considered as defined on $(E, \mathcal{B}_E)$ through the map that imbeds $S_n$ into $E$. Since $\mu_n(E) = \mu_n(S_n) \leq \sum_1^\infty \|H(f_m)\|^p$, some subsequence $\mu_{n_k}$ converges weakly to some measure $\mu$ defined on $(E, \mathcal{B}_E)$. (See [2, p. 427].)

Now for all $n_k \geq m$ and for $t=(t_1, \cdots, t_m)$ we have

$$\int_E |\langle t, s \rangle|^p \, d\mu_{n_k}(s) = \left\| \sum_1^m t_i H(f_i) \right\|^p.$$

Hence by weak convergence we have that

$$\int_E |\langle t, s \rangle|^p \, d\mu(s) = \left\| \sum_1^m t_i H(f_i) \right\|^p.$$

\[\text{License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use}\]
So letting $e_i(s) = s_i$, it follows that the map $\sum_1^m t_i f_i \rightarrow \sum_1^m t_i e_i$ is a linear isometry from the linear span of $\{f_i\}$ into $L^p[E, \mu]$. It follows that the linear isometry can be extended to all of $L$. Now $L^p[E, \mu]$ is separable, hence it itself may be linearly and isometrically imbedded into $L^p[0, 1]$. Q.E.D.

**Remarks.** If $p = 2$ then

$$\int_E \sum_1^\infty (s_m)^2 \, d\mu(s) = \sum_1^\infty \|H(f_m)\|^2$$

and hence letting $n \rightarrow \infty$ we have that

$$\int_E \sum_1^\infty (s_m)^2 \, d\mu(s) = \sum_1^\infty \|H(f_m)\|^2.$$

Now $\mu(E) \leq \sum_1^\infty \|H(f_m)\|^2$ hence $\sum_1^\infty (s_m)^2 = 1$ a.e. with respect to $\mu$. In other words we conclude that $\mu$ lives on the set $S = \{s | s = (s_1, s_2, \cdots) \}$ with $\sum_1^\infty (s_i)^2 = 1$. For $p < 2$ we can make a similar refinement to our theorem, but we leave it for a later paper.

**REFERENCES**


**Department of Mathematics, University of Washington, Seattle, Washington 98105**

**Current address:** Department of Mathematics, Tulane University, New Orleans, Louisiana 70118