ISOMETRIES OF $H^p$ SPACES OF THE TORUS

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Abstract. Denote by $H^p$ ($1 \leq p \leq \infty$) the Banach spaces of complex-valued functions in $L^p$ of the torus whose Fourier coefficients vanish off a half plane determined by a lexicographic ordering. The surjective isometries of the spaces $H^p$ ($p \neq 2$) are characterized in terms of unimodular functions on the circle and conformal maps of the disc. For $1 < p < \infty$ ($p \neq 2$) the proof depends upon a characterization of certain invariant subspaces previously given by the authors.

Let $A$ be the algebra of continuous complex-valued functions on $\{\lambda \in C: |\lambda| = 1\}$ which are uniform limits of polynomials in $\lambda$. Denote by $H^p(d\theta)$ the closure of $A$ in $L^p(d\theta)$ where $d\theta$ denotes normalized Lebesgue measure on the circle (norm closure for $1 \leq p < \infty$; $w^*$ closure for $p = \infty$). It is well known that the Banach spaces $H^p(d\theta)$ may be identified with the Hardy classes by associating with each function in $H^p(d\theta)$ its analytic extension to the open unit disc via the Poisson integral.

DeLeeuw, Rudin, and Wermer [1], and independently Nagasawa [6], characterized the surjective isometries of $H^\infty(d\theta)$ and $H^1(d\theta)$. Forelli [2] extended the characterization to $H^p(d\theta)$ for $1 < p < \infty$, $p \neq 2$. We state their results in Propositions 1 and 2.

Proposition 1. A linear operator $T$ of $H^\infty(d\theta)$ onto $H^\infty(d\theta)$ is an isometry if and only if

\[(1) \quad (Tf)(\lambda) = \alpha f(\tau(\lambda)) \quad (f \in H^\infty(d\theta); \ |\lambda| = 1),\]

where $\alpha$ is a complex constant of modulus 1 and $\tau$ is a conformal map of the unit disc onto itself.
Proposition 2. Let $1 \leq p < \infty$, $p \neq 2$. A linear operator $T$ of $H^p(\mathbb{D})$ onto $H^p(\mathbb{D})$ is an isometry if and only if

$$\tag{2} (Tf)(\lambda) = \alpha(\tau(\lambda))^{1/p} f(\tau(\lambda)) \quad (f \in H^p(\mathbb{D}); \, |\lambda| = 1),$$

where $\alpha$ and $\tau$ are as in Proposition 1.

We denote by $A(T^2)$ the algebra of continuous, complex-valued functions on the torus $T^2 = \{(z, w): |z| = |w| = 1\}$ which are uniform limits of polynomials in $z^m w^n$ where $(m, n) \in A = \{(m, n) : n > 0\} \cup \{(m, 0) : m \geq 0\}$. Denoting normalized Haar measure on $T^2$ by $dm$, we define $H^p$ as the closure of $A(T^2)$ in $L^p(dm)$ (norm closure for $1 \leq p < \infty$; $w^*$ closure for $p = \infty$). The purpose of this paper is to present characterizations of the isometries of $H^p$ onto $H^p$ for $1 \leq p \leq \infty$, $p \neq 2$.

$H^p$ consists of those functions in $L^p(dm)$ whose double Fourier coefficients vanish off the half-plane $\mathcal{S}$ which determines a lexicographic ordering. The maximal ideal space of $A(T^2)$ can be identified with $(\{z: |z| = 1\} \times \{w: |w| \leq 1\}) \cup (\{z: |z| \leq 1\} \times \{0\})$, with $dm$ identified with $(z, w) = (0, 0)$. Since $A(T^2)$ is a logmodular algebra we have at our disposal the results of [4].

We denote by $Z$ and $W$ the functions $Z(z, w) = z$ and $W(z, w) = w$. The closure in $L^p(dm)$ of the polynomials in $Z$ is denoted by $Z^p$; the closure of the polynomials in $Z^m W^n$, $n \geq 1$, by $P$; and finally the closure of the polynomials in $Z$ and $Z$ by $L^p$. By [5, Lemma 5, p. 467],

$$\tag{3} H^p = Z^p \oplus P$$

for $1 \leq p \leq \infty$, where $\oplus$ denotes algebraic direct sum. A function $f$ in $H^p$ is inner if $|f| \equiv 1$; $f$ is outer if $f \cdot A(T^2)$ is dense in $H^p$.

Theorem 1. A linear operator $T$ of $H^\infty$ onto $H^\infty$ is an isometry if and only if

$$\tag{4} (Tf)(z, w) = \alpha f(\tau(z), w \sigma(z)) \quad (f \in H^\infty; \, |z| = |w| = 1),$$

where $\alpha$ is a complex constant of modulus 1, $\tau$ is a conformal map of the unit disc onto itself, and $\sigma$ is a unimodular measurable function.

According to [1, Theorem 3, p. 695] it suffices to prove

Theorem 2. A linear operator $\Psi$ of $H^\infty$ onto $H^\infty$ is an algebra automorphism if and only if

$$\tag{5} (\Psi f)(z, w) = f(\tau(z), w \sigma(z)) \quad (f \in H^\infty; \, |z| = |w| = 1),$$

where $\tau$ and $\sigma$ are as in Theorem 1.
Lemma 1. If $\Psi$ is an algebra automorphism of $H^\infty$, then $\Psi$ carries inner functions to inner functions, $\Psi Z^\infty = Z^\infty$, and $\Psi I^\infty = I^\infty$.

Proof. If $F$ is inner but $\Psi F$ is not, then there exists $\epsilon > 0$ such that $m(K) > 0$ where $K = \{x : |\Psi F(x)| < 1 - \epsilon\}$. Choose $h \in H^\infty$ with $|h(x)| = 1$ on $K$ and $|h(x)| = 1 - \epsilon$ on $T^2 \setminus K$ [4, Theorem 5.9, p. 297]. If $\Psi g = h$, $\|Fg\|_{\infty} = 1$ but $\|\Psi(Fg)\|_{\infty} = \|\Psi F\|_{\infty} \leq 1 - \epsilon$. Thus $\Psi F$ is inner.

Let $M$ be the closure of $\Psi Y$ in $L^2(dm)$. $M$ is clearly invariant under multiplication by functions in $H^\infty$ and also by $\Psi$ where $V = \Psi Z$. For if $f \in I^\infty$, $fZ \in I^\infty$, so $\Psi f = \Psi(fZ) \Psi Z$ or $\Psi(fZ) = (\Psi f)(\Psi Z)$.

If $M$ has the form $FH^2$ for some inner function $F$, then $F \circ \Psi M = FH^2$ so $\Psi \in H^2$. But $\Psi Z \in H^2$ so it is a constant. This contradicts the fact that $\Psi$ is injective, so $M \subseteq I^1$ [4, p. 293]. It follows, using (3), that $\Psi I^\infty \subseteq I^\infty$.

Applying the same argument to the automorphism $\Psi^{-1}$, we conclude that $\Psi Y = Y$.

To show that $\Psi Z^\infty = Z^\infty$, it suffices to show that $\Psi Z \in Z^\infty$. Write $f = \Psi Z$ and suppose $f = f_1 + f_2$ where $f_1 \in Z^\infty$ and $f_2 \in I^\infty$. Then $I^\infty = \Psi(ZI^\infty) = I^\infty$, so $f_2 = fg$ for some $g \in I^\infty$. Thus $g = f_2 f_1 = (f - f_1) f_1 - f_1 f_1$, which is orthogonal to $Z$. Thus $g$ and hence $f_2$ vanish.

Lemma 2. If $E_1 \in Z^\infty$ and $E_2 \in I^\infty$ are inner functions, and if for each Borel set $Y \subseteq T^1$, $m(Y) = m(X)$ where $X = \{(z, w) : (E_1(z), E_2(z, w)) \in Y\}$, then $\mu \ll m$.

Proof. The Fourier-Stieltjes coefficients of $\mu$ are $\hat{\mu}(m, n) = \int E_1^m E_2^n dm$. Thus $\hat{\mu}(m, 0) = a^m$ for $m \geq 0$ ($a = \int E_1 dm$). Since $E_2 \in I^\infty$, $\mu(m, n) = 0$ for $n \geq 1$. It follows that $\mu$ is the product measure $\mu = P dz \times dw = Q dm$, where $P(z) = (1 - |a|^2)/(|1 - az|^2$, $dz$ and $dw$ are each Lebesgue measure, and $Q \subseteq L^\infty(dm)$. In particular, if $Y$ is $m$-null, then $X$ is $m$-null. This argument is based on Forelli [2, p. 724].

Proof of Theorem 2. By Lemma 1, $\Psi W = W$. In fact $\Psi W = W\sigma$ for some $\sigma(z) = \sigma(z, w) \in L^\infty$, as can be shown by an argument similar to that by which we showed $\Psi Z \in Z^\infty$. Writing $\tau(z) = \tau(z, w) = (\Psi Z)(z, w)$, we see that $\tau$ is a conformal map of the disc by Proposition 1. Setting $E_1 = \tau$ and $E_2 = w\sigma$ in Lemma 2, we conclude that $f(\tau, w\sigma)$ is well defined for all measurable functions $f$. Thus (5) holds for all $f$ in the algebra $\mathcal{A}$ generated by $Z^n W^n$, $m, n \in \mathcal{S}$.

To establish (5) for all $f \in H^\infty$, it suffices to show that the automorphism $\Phi(f) = \Psi^{-1}(f(\tau, w\sigma))$ is the identity. We have seen that $\Phi Z = Z$ and $\Phi W = W$ and the proof of Proposition 1 shows that $\Phi$ is the identity on $Z^\infty$. Thus it suffices to show that $\Phi$ is the identity on $I^\infty$.

First we show that $\Phi(\chi_K W) = \chi_K W$ where $\chi_K$ is a characteristic function in $L^\infty$. Since the function $\Phi(\chi_K W)/W$ is equal to its own square, it too is a characteristic function $\chi_K \in L^\infty$. There remains only to show that $K = K'$,
or in fact that \( K \subseteq K' \) since the argument also applies to \( \Phi^{-1} \). If not, there exists a nonzero continuous function \( f \in L^\infty \) with zero set \( K_1 \subseteq K/K' \) of positive measure. Then

\[
(6) \quad 0 = \Phi(fW)\Phi(\chi_{K_1}W) = fW\chi_{K_1'}W.
\]

Since \( K_1 \subseteq K \), \( K_1' \subseteq K' \), so \( f \) does not vanish on \( K_1' \). This contradicts (6).

Thus \( \Phi(\chi_KW) = \chi_KW \), and in general for \( g \in L^\infty \), \( \Phi(gW^n) = gW^n \) (\( n \geq 1 \)). If \( g \in L^\infty \), \( g = \sum_{i=1}^n g_iW^i + hW^n \) where \( g_i \in L^\infty \) and \( h \in L^\infty \). Since \( \Phi g = \sum_{i=1}^n g_iW^i + (\Phi h)W^n \) where \( \Phi h \in L^\infty \), the Fourier coefficients of \( g \) and \( \Phi g \) agree, so \( \Phi g = g \).

\textbf{Remark.} Using essentially the same argument we can show that the automorphisms of \( A(T^2) \) are also given by (5) except that here \( \sigma \) is continuous. However, this can more easily be done by considering the homeomorphisms of the maximal ideal space of \( A(T^2) \) induced by the automorphisms of the algebra.

\textbf{Theorem 3.} \( A \) linear operator \( T \) of \( H^p \) onto \( H^p \) (\( 1 \leq p < \infty \), \( p \neq 2 \)) is an isometry if and only if

\[
(7) \quad (Tf)(z, w) = \alpha(\tau(z))^{1/p} \cdot f(\tau(z), w\sigma(z)),
\]

for all \( f \in H^p \), where \( |z| = |w| = 1 \), \( \alpha \) is a complex constant of modulus 1, \( \tau \) is a conformal map of the unit disc onto itself, and \( \sigma \) is a unimodular measurable function on the circle.

The proof depends on our results in [5] on the characterization of sesqui-invariant subspaces of \( H^p \). A closed subspace \( M \subseteq H^p \) is called \textit{invariant} if \( fM \subseteq M \) for all \( f \in H^\infty \). An invariant subspace \( M \subseteq H^p \) is called \textit{sesqui-invariant} if \( ZM \subseteq M \) and \textit{simply invariant} if this is not the case. If \( M \) is sesqui-invariant, it has the form

\[
M = \chi_E \cdot \psi \cdot I^p
\]

where \( \psi \) is unimodular and \( \chi_E \) is the characteristic function of the support set of \( M \) ([5, Theorem 3, p. 471]; see also [5, p. 473 for the torus case]). If \( M \) is simply invariant, it has the form \( M = \psi H^p \) (\( \psi \) unimodular) by the generalized Beurling theorem [8].

\textbf{Lemma 3.} \( \text{Let } F = T(1) \text{ and } E \text{ be the support set of } F. \text{ Then } m(E) = 1. \)

\textbf{Proof.} Since \( F \in H^p \), \( \chi_E \) is independent of \( w \), so \( G = w(1 - \chi_E) \in H^p \). Let \( g = T^{-1}(G) \). Thus

\[
\int |1 \pm g|^p \, dm = \int |F \pm G|^p \, dm
= \int |F|^p \, dm + \int |G|^p \, dm = 1 + \int |g|^p \, dm.
\]
Therefore
\[ \int |1 + g|^p \, dm + \int |1 - g|^p \, dm = 2 \left[ 1 + \int |g|^p \, dm \right]. \]

By [7, p. 275], \( g = 0 \) a.e., so \( m(E) = 1 \).

**Proof of Theorem 3.** Lemma 3 insures that \( dv = |F|^p \, dm \) and \( dm \) are mutually absolutely continuous. Using Forelli's argument [2, Proposition 2, p. 723] it follows that \( Sf = TfF \) defines an isometry \( S \) of \( H^p \) into \( L^p(dv) \) which takes the algebra \( \mathcal{A} \) generated by \( Z^mW^n, (m, n) \in \mathcal{S} \), multiplicatively into \( L^\infty(dv) \). Write \( E_1 = S(Z) \) and \( E_2 = S(W) \). For \( f \in \mathcal{A} \), we have

\[ Tf(z, w) = F \cdot f(E_1, E_2). \]

We show that \( E_1 \in L^\infty \) and \( E_2 \in L^\infty \). Since \( F \in H^p \), the sesqui-invariant subspace generated by \( F \) has the form \( JIp \), where \( J \) is unimodular. Thus \( F = JG \) where \( G \in Ip \) and the sesqui-invariant subspace generated by \( G \) is \( Ip \). For \( f \in S(\mathcal{A}) \), \( WF \in Ip \), and the property of \( G \) insures that \( W^2f \in Ip \). Thus the invariant subspace generated by \( S(\mathcal{A}) \) has the form \( \psi Ip \) or \( \psi H^p \), \( \psi \) unimodular.

In the first case \( f, W \in \psi Ip \), so \( f \in L^\infty \oplus Ip \) for all \( f \in S(\mathcal{A}) \) and similarly for the second case. In particular \( E_1 \in Ip \) and \( E_2 \in L^\infty \). The same argument applied to the algebra generated by \( Z^mW^n, n \geq 1 \), shows that \( E_1 \in L^\infty \oplus Ip \), so \( E_1 \in L^\infty \).

We conclude that \( F \notin Ip \) (otherwise \( T \) would map \( H^p \) onto \( Ip \)). Thus \( \int \log |F| \, dm > -\infty \), so \( F = JG \) where now \( J \) is inner and \( G \) is outer. Also \( \{ FE_n^m \}, m \geq 0 \), generate a simply invariant subspace, so by the usual argument \( E_1 \in Z^\infty \). Since \( G \) is outer, the invariant subspace \( N \) generated by \( \{ JE_1^mE_2^m \}, n > 0 \), is contained in \( H^p \). Since \( N = \psi H^p \) would imply that \( E_1 \in H^p \), we have \( N \subseteq Ip \) so \( JE_1 \in Ip \). \( J \in H^p \) but \( J \notin Ip \) (because \( F \notin Ip \)), so \( E_2 \in L^\infty \). Thus \( T \) takes \( Ip \) into \( Ip \).

Thus using Lemma 2, \( f(E_1, E_2) \) is well defined for all measurable functions \( f \). The density of \( \mathcal{A} \) in \( H^p \), \( 1 \leq p < \infty \), insures that (8) holds for all \( f \in H^p \). Imitating Forelli's argument [2, p. 726] one shows that the function \( Q \) constructed in the proof of Lemma 2 satisfies

\[ \int_X |F|^p \, dm = \int_X 1/Q(E_1) \, dm \]

for all Borel sets \( X \subseteq T^2 \). Since \( T \) is surjective, both \( T \) and \( T^{-1} \) carry \( Ip \) into \( Ip \), so that \( TT^p = Ip \). Again using the argument of [2] (beginning at the bottom of p. 726) it follows that \( E_1 \), considered as a function of \( z \) alone, is a.e. the boundary value function of a conformal map \( \tau_1 \) of the disc onto itself. Define \( \tau(z, w) = \tau_1(z) \). We have \( |\tau'| = 1/Q(\tau) \) and (9) becomes
for all Borel sets \( X \subseteq \mathbb{T}^2 \). Thus \( F \) and \( (\tau')^{1/p} \) have the same modulus. Since the latter is outer, we can show that they differ by a constant of modulus one by showing \( F \) is outer. If \( F = JG, J \) inner and \( G \) outer, then \( GH^p = H^p = TH^p = Fh^p \). Dividing by \( G \), \( H^p = Jh^p \), so \( J \equiv 1, |z| = 1 \).

To complete the proof it suffices to show that \( E_\sigma = W\sigma \) where \( \sigma \in L^\infty \). For this we need to show that \( \sigma h^p = l^p \) (see the analogous argument for \( p = \infty \)). But since \( F \) and \( 1/F \) are bounded, \( \sigma h^p = \overline{W}(SW)(Sl^p) = \overline{W}(SL^p) = l^p \).

For the case \( p = 1 \), Theorem 3 can also be obtained by adapting the original argument of deLeeuw, Rudin and Wermer [1, Theorem 2, p. 694] in which they deduce the isometries of \( H^1(d\theta) \) by exploiting the special properties of the extreme points of the unit ball of \( H^1(d\theta) \). To do this one needs three facts about functions on the torus: (a) the extreme points of the unit ball of \( H^1 \) are the outer functions of norm one (Gamelin [3]), (b) the identity \( \int f dm = \int (f, \overline{\omega}) \eta^\prime dm \) (a straightforward calculation), and (c) the result of Lemma 4 below. Let \( B^e \) be the set of extreme points in the unit ball of \( H^1 \), \( P(m) = \{z: |z| < 1\} \times \{0\} \), and \( D_z = \{z\} \times \{w: |w| < 1\} \) for each \( |z| = 1 \).

**Lemma 4.** A function \( f \in H^1 \) of norm 1 lies in the closure of \( B^e \) if and only if
\[
(10) \quad \text{\( f \) has no zeros on \( P(m) \) and \( \hat{f} \) has no zeros on \( D_z \) for almost all \( z \).}
\]

**Proof.** If \( f \) lies in the closure of \( B^e \), then there exist \( f_n \in B^e \) converging uniformly on compact sets to \( f \) on \( P(m) \) and on each \( D_z \) for almost all \( z \). (10) follows.

Conversely suppose (10) holds. Define \( f_r(z, w) = f(z, rw), 0 < r < 1 \). Let \( f_{r_*} = F_r g_r, F_r \) inner, \( g_r \) outer. One shows that \( F_r \) is independent of \( r \), say \( F_r = F \in \mathbb{Z}^\infty \). Let \( h_r(z, w) = F(rz) \). Then \( f \) is the \( L^1 \) limit of the outer functions \( h_r g_r \), so \( f \) lies in the closure of \( B^e \).

**References**


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