STONE'S THEOREM FOR A GROUP OF UNITARY OPERATORS OVER A HILBERT SPACE

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ABSTRACT. The spectral representation for a group of unitary operators acting on a Hilbert space where the parameter set is a separable real Hilbert space is obtained. The usual spectral representation of such a group of unitary operators is when the parameter set is a locally compact abelian group (Stone's theorem). The main result used in the proof is the Bochner theorem on the representation of positive definite functions on a real Hilbert space.

Introduction. This paper discusses a spectral representation theorem for a group of unitary operators acting on a Hilbert space $H$, where the parameter set is a real Hilbert space $T$. The usual spectral representation of such a group of unitary operators (the well-known Stone theorem) is when the set $T$ is a locally compact abelian group [3, p. 392]. The main result used in the course of our work is the Bochner theorem on the representation of positive definite functions on a real Hilbert space $T$. This theorem is included in the interesting work of L. Gross [2] in Harmonic analysis on Hilbert space.

1. Preliminaries. We begin this section by introducing some notation which will be used later. $T$ will be a real Hilbert space. We will use $s$, $t$, etc. for points in $T$. By a Borel set in $T$ we shall mean a set in the $\sigma$-algebra determined by the collection of open sets in $T$. In $T$ the inner product and norm will be denoted by $(\cdot, \cdot)_T$ and $|\cdot|_T$. A complex-valued function $\varphi$ on $T$ will be called positive definite if for every finite set $t_1, \cdots, t_n$ of points in $T$ the matrix $(a_{ij})$ defined by $a_{ij} = \varphi(t_i - t_j)$ is nonnegative definite. The following definition is due to L. Gross [2, p. 5].

1.1 Definition (topology $\tau$ for $T$). The topology of $\tau$ is defined as the weakest topology for which all Hilbert-Schmidt operators on $F$ into $T$ are continuous from $\tau$ to the strong topology of $T$.

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In these terms we can state Bochner’s theorem as proved by L. Gross [2, p. 20], as follows:

1.2 Theorem. A complex-valued function \( \varphi \) on \( T \) is positive definite and continuous with respect to the \( \tau \)-topology if and only if

\[
\varphi(t) = \int_T e^{it(s)} \mu(ds),
\]

where \( \mu \) is a positive finite measure on \( T \).

It is known that if \( T \) is separable then \( \varphi \) determines \( \mu \) uniquely (cf. [2, p. 1 and p. 6]). Since a complex measure \( \mu \) can be decomposed into 
\[ \mu = \mu_1 - \mu_2 + i(\mu_3 - \mu_4), \]
where \( \mu_1, \mu_2, \mu_3, \mu_4 \) are nonnegative finite measures, we can state the following corollary:

1.4 Corollary. If \( \psi \) is any complex-valued function defined on a separable real Hilbert space \( T \) which has the form

\[
\psi(t) = \int_T e^{it(s)} \, d\mu,
\]

when \( \mu \) is a complex-valued measure, then \( \mu \) is uniquely determined by \( \psi \).

The groups of unitary operators whose spectral representation we shall establish will be denoted by \( \{U_t, t \in T\} \). Precisely speaking we have a complex Hilbert space \( H \) {with inner product \((\cdot, \cdot)_H\) and norm \(|\cdot|_H\)} and for each \( t \in T \), we have a unitary operator \( U_t \) acting on \( H \) such that \( U_t U_s = U_{t+s} \) for all \( s, t \in T \). Points of \( H \) will be denoted by \( x, y, \) etc. We shall assume in proving our spectral theorem that \( U_t \) is weakly \( \tau \)-continuous in \( t \), i.e., 
\((U_t x, y)\) is continuous with respect to the \( \tau \)-topology for each \( x, y \in H \). We remark that our proof of Theorem 2 uses the representation of positive definite \( \tau \)-continuous functions \( \varphi \) [1, p. 20], in the same spirit as the corresponding Bochner theorem concerning positive definite functions on a locally compact abelian group was used to obtain the Stone’s theorem for a group of unitary operators over a locally compact abelian group.

2. The spectral theorem. By a spectral measure on \( T \) for a given Hilbert space \( H \) we shall mean a family of operators \( \{E(B)\} \) on \( H \) into \( H \) on \( \mathcal{B}(T) \), the Borel sets in \( T \) with the following properties:

(a) \( E(B) \) is an orthogonal projection operator.
(b) \( E(\emptyset) = 0 \).
(2.1) (c) \( E(T) = I \).
(d) If \( B = \bigcup_i B_i \), when \( B_i \) are disjoint sets in \( \mathcal{B}(T) \), then \( E(B) = \sum_i E(B_i) \).

We can now state and prove our spectral theorem.
2.2 Theorem. Let \{U_t, t \in T\} be a weakly \(r\)-continuous group of unitary operators on a given Hilbert space \(H\), over a separable Hilbert space \(T\). Then there exists a unique spectral measure \(\{E(B), B \in \mathcal{B}(T)\}\) such that

\[
U_t = \int_T e^{it \xi} dE(\xi), \quad t \in T.
\]

A bounded operator commutes with every \(U_t\) if and only if it commutes with each \(E(B), B \in \mathcal{B}(T)\).

Proof. For each \(x \in H\), the function \((U_t x, x)\) is a positive definite function. Since by assumption \((U_t x, x)\) is \(r\)-continuous, by Theorem 1.2 and (1.3),

\[
(U_t x, x) = \int_T e^{it \xi} \mu_\xi(ds).
\]

From (2.4) it follows that for each \(x, y \in H\) there exists a complex-valued measure \(\mu_{x,y}\) such that

\[
(U_t x, y) = \int_T e^{it \xi} \mu_{x,y}(ds).
\]

The uniqueness of this representation (Corollary 1.4) then shows that for each \(B \in \mathcal{B}(T)\), \(\mu_{x,y}(B)\) is a bilinear functional and also

\[
\mu_{x,y}(B) = \overline{\mu_{y,x}(B)}.
\]

We note that \(|\mu_{x,y}(B)|^2 \leq \mu_{x,z}(B) \mu_{y,v}(B)\). Putting \(t = 0\) in (2.4) we get that

\[
\mu_{x,z}(B) \leq |x|^2;
\]

then (2.6) and (2.7) imply that \(\mu_{x,z}(B)\) is a bounded bilinear functional on \(H\). Hence by [3, p. 202], we conclude that for each Borel set \(B \in \mathcal{B}(T)\) there is a bounded operator \(E(B)\) on \(H\) into \(H\) such that \((E(B)x, y) = \mu_{x,y}(B), E(B) = E^*(B)\). If \(B_1, B_2 \in \mathcal{B}(T)\) we have

\[
(U_{t+s} x, y) = \int_T e^{i(s \xi)}(E(d\xi)x, y)
\]

\[
= \int_T e^{i(t \xi)}e^{i(s \xi)}(E(d\xi)x, y) = \int_T e^{i(t \xi)}\nu(d\xi),
\]

when \(\nu(d\xi) = e^{i(s \xi)}(E(d\xi)x, y)\). We also have

\[
(U_{t+s} x, y) = \int_T e^{i(s \xi)}(E(d\xi)U_s x, y).
\]

By the uniqueness result (Corollary 1.4), we have that, for each \(B_1 \in \mathcal{B}(T)\),

\[
\int e^{i(s \xi)}(E((d\xi) \cap B_1)x, y) = \int_{B_1} e^{i(s \xi)}(E(d\xi)x, y) = (E(B_1)U_s x, y).
\]
Also we have

\[(2.9) \quad (E(B_1)U_\sigma x, y) = (U_\sigma x, E(B_1)y) = \int e^{it_\xi \sigma}(E(d\xi)x, E(B_1)y).\]

From (2.8) and (2.9) we obtain \((E(B \cap B_1)x, y) = (E(B)x, E(B_1)y)\). Therefore \(E(B \cap B_1) = E(B)E(B_1)\). Since

\[(U_\sigma x, y) = (x, y) = \int \mu_{x,y}(ds) = \mu_{x,y}(T)\]

we get that, for all \(x, y \in H\), \((E(T)x, y) = (x, y)\); hence \(E(T) = I\). Let \(B = \bigcup B_i\), \(B_i\)'s disjoint in \(\mathcal{B}(T)\). Then

\[(E(B)x, y) = \mu_{x,y}(B) = \mu_{x,y}(\bigcup B_i) = \sum_i \mu_{x,y}(B_i) = \sum (E(B_i)x, y).\]

Hence, \(E(B) = \sum_i E(B_i)\).

This implies \(E(\emptyset) = 2E(\emptyset)\) and hence \(E(\emptyset) = 0\). So that \(E\) is a spectral measure. The existence of the integral \(\int_T e^{it_\xi \sigma}E(d\xi)\) which we call \(V_t\) is well known. Then from the usual operational calculus it follows that \((V_t x, y) = \int e^{it_\xi \sigma}(dE(w)x, y)\), so that if each \(t \in T\) we have \((V_t x, y) = (U_t x, y)\), and hence \(V_t = U_t\).

In view of Corollary 1.4 the uniqueness of \(E(\cdot)\) is clear.

Now let \(V\) be any bounded linear operator on \(H\) into \(H\). An argument similar to the one given in [1, p. 593] shows that \(V\) commutes with \(U_t\), \(t \in T\), if and only if \(V\) commutes with \(E(B), B \in \mathcal{B}(T)\).

3. Application to stochastic processes. Let \((\Omega, B, P)\) be a probability space and let \(\{X_t\}\) be a 2nd order stochastic process with \(t \in T\) (\(T\) is a separable real Hilbert space). \(\mathcal{E}\) will denote the expectation operator. If \(\mathcal{E}(X_t \bar{X}_s)\) depends only on the difference of \(t - s\) then \(\{X_t\}\) is said to be a stationary stochastic process. The theory of such processes over any locally compact abelian group has been studied [4]. The same line of proof as in [4] can be used to prove:

3.1 Theorem. (i) Let \(\{X_t, t \in T\}\) be a stationary stochastic process with values in \(L_2(\Omega, B, P)\).

(ii) Let \(H = \text{closed subspace in } L_2(\Omega, B, P)\) spanned by \(\{X_t, t \in T\}\).

(iii) Let \(\{X_t\}\) be \(\tau\)-continuous, i.e. the correlation function \(\mathcal{E}(X_t \bar{X}_0)\) is continuous with respect to the \(\tau\)-topology of \(T\).

Then

(a) There exists a unique spectral measure \(E\) on \(\mathcal{B}(T)\) for the Hilbert space \(H\) such that

\[(3.2) \quad X_t = \int_T e^{it_\xi \sigma}E(d\xi)X_0, \quad \mathcal{E}(X_t \bar{X}_s) = \int_T e^{i(t-s)_\xi}(E(d\xi)X_0, X_0).\]
The measure \( dF = (E(d\xi)X_0, X_0) \) is called the spectral distribution of the process \( \{X_t, t \in T\} \).

(b) The mapping \( e^{it\cdot} \rightarrow X_t \) is an isometry on \( L_{2,F} \) onto \( H \).

Proof of (b) depends on the fact that \( e^{it\cdot} \) are dense in \( L_{2,F}(T) \) because of Corollary 1.4.

Remark. Theorem 3.1 and representation (3.2) can be used in the prediction theory of stationary stochastic processes over a separable Hilbert space to obtain results similar to the usual work in the prediction of stationary stochastic processes over a locally compact abelian group [4]. This work is being investigated by the author and the result will be published elsewhere.

References


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