EQUIVALENT TYPES OF INVARIANT MEANS ON LOCALLY COMPACT GROUPS

P. F. RENAUD

Abstract. For $G$ a locally compact amenable group, we establish the equivalence of left invariant means and topologically left invariant means on $L^\infty(G)$.

1. Introduction and notation. Let $G$ be a locally compact group with left Haar measure $\mu$. Let $L^1(G)$ and $L^\infty(G)$ denote the usual Banach function spaces on $G$. $L^1(G)$ is a Banach $\ast$-algebra under the convolution operation

$$x \ast y(g) = \int x(h)y(h^{-1}g) \, d\mu(h)$$

and the adjoint map

$$x^\ast(g) = \Delta(g^{-1})x(g^{-1})$$

where $\Delta$ is the modular function on $G$. A weight on $G$ is a nonnegative function $x \in L^1(G)$ such that $\int x(g) \, d\mu(g) = 1$. Denote by $P$ the set of all weights on $G$ and observe that $P$ is a semigroup under convolution. For $f$ a complex-valued function on $G$, define $\phi_f(g) \in G$ by $\phi_f(h) = f(g^{-1}h)$.

A linear functional $m$ on $L^\infty(G)$ is called a mean if

(i) $m(f) \geq 0$ for all $f \in L^\infty(G)$, $f \geq 0$ and

(ii) $m(1) = 1$ where $1$ denotes the identity function.

Clearly every weight $x$ in $P$ defines a mean on $L^\infty(G)$ via $m(x) = \int x(g) f(g) \, d\mu(g)$.

Let $m$ be a mean on $L^\infty(G)$. $m$ is called a left invariant mean (LIM) if

$$m_x(f) = m(f) \quad \text{for all } f \in L^\infty(G), \, g \in G.$$

$m$ is called a topologically left invariant mean (TLIM) if

$$m(x \ast f) = m(f) \quad \text{for all } f \in L^\infty(G), \, x \in P,$$

or equivalently

$$m(x \ast f) = m(f) \int x(g) \, d\mu(g) \quad \text{for all } f \in L^\infty(G), \, x \in L^1(G).$$

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G is called amenable if there exists a \( L^\infty(G) \). TLIM's were introduced by Hulanicki [4] as a natural extension of LIM's to non-discrete groups. Among other results he proved that every TLIM is also a LIM. Subsequently, Namioka [5] showed that the existence of a LIM implies the existence of a TLIM. The purpose of this note is to show that in fact every LIM is also a TLIM. This answers a question raised by Greenleaf (see [3, Lemma 2.2.2 and remarks]).

2. Equivalence of LIM's and TLIM's. We shall now prove the following

**Theorem.** Let \( G \) be an amenable group, \( m \) a LIM on \( L^\infty(G) \). Then \( m \) is a TLIM.

For \( f \in L^\infty(G) \) and \( x \in L^1(G) \), left invariance of \( m \) gives

\[
m((\varphi \ast g) \ast f) = m((\varphi \ast f)) = m(\varphi \ast f) \quad \text{for all } g \in G.
\]

Hence \( x \to m(\varphi \ast f) \) is a left invariant bounded linear functional on \( L^1(G) \) so that there exists a constant \( k(f) \) such that

\[
m(\varphi \ast f) = k(f) \int x(g) \, d\mu(g) \quad \text{for all } x \in L^1(G).
\]

It is immediate that \( k \) is a mean on \( L^\infty(G) \). Further if \( x \in \mathcal{P} \) then \( x \ast x \in \mathcal{P} \) so that

\[
k(x \ast f) = m(x \ast (x \ast f)) = m((x \ast x) \ast f) = k(f)
\]

and \( k \) is a TLIM. The theorem will be proved if we show that \( m = k \).

Fix \( f \in L^\infty(G) \). Choose a net \( \{x_\gamma\}_{\gamma \in \Omega} \subseteq \mathcal{P} \) such that \( w^* \lim \gamma x_\gamma = m \) and define \( F_\gamma \) on \( G \) by \( F_\gamma(g) = \langle \varphi, x_\gamma, f \rangle \). Left invariance of \( m \) means that \( F_\gamma \to m(\varphi) \) pointwise on \( G \). The theorem will follow from the following

**Lemma.** \( F_\gamma \to m(\varphi) \) almost uniformly on every compact subset of \( G \).

**Proof.** If we were dealing with sequences rather than nets, then the lemma would be a trivial application of Egoroff's theorem. With nets, however, a little delicacy is required.

Let \( K \) be a compact set with \( \mu(K) > 0 \). For \( k \) a positive integer, \( \gamma \in \Omega \), define

\[
E_{k, \gamma} = \bigcap_{\gamma' \geq \gamma} \{ g \in K : |F_{\gamma'}(g) - m(\varphi)| \leq 1/k \}.
\]

Since \( F_\gamma \) is continuous, \( E_{k, \gamma} \) is a compact subset of \( K \). Note that for fixed \( k \), \( \{E_{k, \gamma}\} \) is an increasing net (in the sense that \( \gamma \geq \gamma' \Rightarrow E_{k, \gamma} \supseteq E_{k, \gamma'} \)) with \( \bigcup_{\gamma} E_{k, \gamma} = K \). Let \( \chi_K, \chi_{E_{k, \gamma}} \) be the characteristic functions of \( K \) and \( E_{k, \gamma} \), respectively. We then have that \( \{\chi_{E_{k, \gamma}}\} \) is a bounded monotone increasing net in \( L^\infty(G) \) for each \( k \) and \( \chi_K = \sup_{\gamma} \chi_{E_{k, \gamma}} \). Now \( L^\infty(K) \) may be regarded
as a $W^*$-algebra on the Hilbert space $L^2(K)$. The predual of $L^\infty(K)$ is $L^1(K)$ so that every nonnegative element in $L^1(K)$ is a normal positive linear functional on $L^\infty(K)$ [1, Chapitre 1, §§3 and 4]. Hence $\langle \chi_K, \chi_K \rangle = \sup_\gamma \langle \chi_K, \chi_{E_{k, \gamma}} \rangle$ or,
\[
\lim_{\gamma} \mu(E_{k, \gamma}) = \mu(K) \quad \text{for each } k.
\]
Fix $\varepsilon > 0$. For each $k$, choose $\gamma_k$ such that $\mu(K \setminus E_{k, \gamma_k}) < \varepsilon/2^k$ and let $E_0 = \bigcap_k E_{k, \gamma_k}$. $E_0$ is a compact set and
\[
\mu(K \setminus E_0) = \mu\left( \bigcup_k K \setminus E_{k, \gamma_k} \right) \leq \sum_k \mu(K \setminus E_{k, \gamma_k}) < \varepsilon.
\]
Finally it is clear that $F_\gamma \to m(f)$ uniformly on $E_0$.

It should be noted that the above theorem resembles somewhat the condition (FC*) of [2]. Using a technique similar to the one employed in the proof of Lemma 1.4.3 of [2], we could show directly that the above lemma implies that $F_\gamma \to m(f)$ uniformly on all compact sets.

**Proof of Theorem.** By the above lemma we can find a compact set $E$ with $\mu(E) > 0$ such that $F_\gamma \to m(f)$ uniformly on $E$. Therefore
\[
\lim_{\gamma} \int_E F_\gamma(g) \, d\mu(g) = m(f)\mu(E).
\]
But
\[
\int_E F_\gamma(g) \, d\mu(g) = \int_G \chi_E(g) \left[ \int_G x_\gamma(g^{-1}h)f(h) \, d\mu(h) \right] \, d\mu(g)
\]
\[
= \int_G (\chi_E * x_\gamma)(h)f(h) \, d\mu(h)
\]
\[
= \langle \chi_E * x_\gamma, f \rangle = \langle x_\gamma, \chi_E^* * f \rangle
\]
so that $\lim_{\gamma} \int_E F_\gamma(g) \, d\mu(g) = m(\chi_E^* * f)$. Hence
\[
m(f)\mu(E) = m(\chi_E^* * f)
\]
\[
= k(f) \int \chi_{E_0}^*(g) \, d\mu(g) = k(f)\mu(E).
\]
Therefore $m(f) = k(f)$ and $f$ being arbitrary, $m = k$. Hence $m$ is a TLIM.

The notion of LIM and TLIM may be applied to $CB(G)$—the space of bounded continuous functions on $G$. The method above may be applied here to show that on $CB(G)$ every LIM is again a TLIM.

**References**


DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CANTERBURY, CHRISTCHURCH, NEW ZEALAND