MULTIPLIER OPERATORS ON $B^*$-ALGEBRAS

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Abstract. The purpose of this paper is to give a characterization of the dual $B^*$-algebra and the algebra of bounded linear operators on Hilbert space in terms of their multipliers.

1. All algebras and vector spaces under consideration are over the complex field $\mathbb{C}$. If $A$ is a Banach algebra, $A^*$ will denote the first conjugate space and $A^{**}$ the second conjugate space of $A$. For any Hilbert space $H$, $L(H)$ will denote the algebra of continuous linear operators on $H$.

Let $A$ be a $B^*$-algebra. Following Máté [5], we call a bounded linear operator $T$ mapping $A$ into itself a multiplier if $T(xy) = x(Ty)$ for all $x, y \in A$. The set $M(A)$ of all multipliers on $A$ is a Banach algebra. For every $a \in A$, the right multiplication operator $T_a$ is a multiplier on $A$, $I_A = \{T_a: a \in A\}$ is a closed left ideal of $M(A)$ and the mapping $a \mapsto T_a$ is an isometric anti-isomorphism of $A$ onto $I_A$.

From [3, p. 869, Theorem 7.1] it follows that if $A$ is a $B^*$-algebra then the two Arens products defined on $A^{**}$ coincide. For later use we sketch one of the Arens products. We do this in stages as follows [1], [3]: Let $x, y \in A$, $f \in A^*$, $F, G \in A^{**}$.

(i) Define $f \circ x$ by $(f \circ x)y = f(xy); f \circ x \in A^*$.

(ii) Define $G \circ f$ by $(G \circ f)x = G(f \circ x); G \circ f \in A^*$.

(iii) Define $F \circ G$ by $(F \circ G)f = F(G \circ f); F \circ G \in A^{**}$.

For any Hilbert space $H$, $L_C(H)$ will denote the subalgebra of $L(H)$ consisting of compact operators and $\tau_C(H)$ the subalgebra of trace class operators on $H$. We shall denote the trace function of $\tau_C(H)$ by $\text{tr}()$ and the trace norm by $\tau(); \tau(T) = \text{tr}((T^*T)^{1/2})$ for all $T \in \tau_C(H)$. As a Banach space $\tau_C(H)$ can be identified with the conjugate space of $L_C(H)$ in the following way: For each continuous linear functional $f$ on $L_C(H)$ there exists a unique $T$ in $\tau_C(H)$ such that $f(S) = \text{tr}(ST)$ for $S \in L_C(H)$ and $\|f\| = \tau(T)$ [7, p. 46, Theorem 1]. Similarly, the conjugate space of $\tau_C(H)$ can be identified (isometrically) with $L(H)$ [7, p. 47, Theorem 2].
Thus the second conjugate space of $LC(H)$ is isometrically isomorphic to $L(H)$. In fact, it can be shown that this isomorphism is actually a $*$-isomorphism when the second conjugate space of $LC(H)$ is given the Arens product.

Let $\{A_\lambda : \lambda \in \Lambda \}$ be a family of Banach algebras. Let $\sum A_\lambda$ be the set of all functions on $\Lambda$ with $f(\lambda) \in A_\lambda$, for each $\lambda$, and such that $\|f\| = \sup_\lambda \|f(\lambda)\| < \infty$. Then under the usual operations for functions and the norm $\|f\|$, $\sum A_\lambda$ is a Banach algebra. It is called the normed full direct sum of the algebras $A_\lambda$ [6, p. 77]. Let $(\sum A_\lambda)_0$ be the subset of $\sum A_\lambda$ consisting of all $f$ such that, for every $\epsilon > 0$, the set $\{\lambda: \|f(\lambda)\| \geq \epsilon\}$ is finite. Then $(\sum A_\lambda)_0$ is a closed subalgebra of $\sum A_\lambda$ [6, p. 107].

For any set $S$ in a Banach algebra $A$, let $l(S)$ and $r(S)$ be the left and right annihilators of $S$, respectively. $A$ is called dual if $l(r(J)) = J$ and $r(l(R)) = R$ for every closed left ideal $J$ and every closed right ideal $R$ of $A$. As usual $cl(S)$ will denote the closure of $S$ in $A$.

2. We devote this section to several lemmas which will be useful to us in §3.

**Lemma 2.1.** To each multiplier $T$ on the algebra $LC(H)$ there corresponds a unique element $a_T$ in $L(H)$ such that $T(s) = sa_T$ for all $s \in LC(H)$; $\|T\| = \|a_T\|$. Thus the mapping $T \rightarrow a_T$ is an isometric anti-isomorphism of $M(LC(H))$ onto $L(H)$.

**Proof.** Let $A = LC(H)$ and let $T \in M(A)$. Since $A$ has an approximate identity, by [5, p. 810, Theorem 1], there exists a unique element $F$ in $A^{**}$ such that

$$
(F \circ f)s = f(T(s)) \quad (s \in A, f \in A^*) \quad (1)
$$

For $s \in A$ and $f \in A^*$, let $t_f$ and $t_{f_s}$ be the elements of $\tau c(H)$ such that $f(a) = tr(at_f)$ and $(f \circ s)(a) = tr(at_{f_s})$ for all $a \in A$. (See [7, p. 46, Theorem 1].) Since

$$
tr(t_{f_s}a) = f(sa) = (f \circ s)a = tr(t_{f_s}a) \quad (a \in A),
$$

[7, p. 45, Lemma 1] shows that

$$
t_{f_s} = t_f \quad (s \in A, f \in A^*) \quad (2)
$$

Thus

$$
(F \circ f)s = F(f \circ s) = tr(t_{f_s}t_F) = tr(t_{f_s}t_F) \quad (s \in A, f \in A^*) \quad (3)
$$

where $t_F$ is the unique element in $L(H)$ such that $F(f) = tr(t_F)$ for all $f \in A^*$. (See [7, p. 47, Theorem 2].) But $f(T(s)) = tr(t_F(T(s)))$. Hence from (1) and (3) it follows that

$$
f(T(s)) = tr(t_F(T(s)) \quad (f \in A^*).$$
Recalling [7, p. 45, Lemma 1], we see that \( T(s) = st_F \) for all \( s \in A \). Taking \( a_T = t_F \) completes the proof.

**Corollary 2.2.** Let \( A = L^C(H) \). Then there exists an isometric anti-isomorphism \( \phi \) of \( M(A) \) onto \( A^{**} \) such that \( \phi(I_A) = \pi(A) \), where \( \pi(A) \) is the canonical image of \( A \) in \( A^{**} \).

**Proof.** For each \( a \in A \), the right multiplication operator \( T_a \) is a multiplier on \( A \). Hence by Lemma 2.1, there exists \( b \in L(H) \) such that \( T_a x = xb \) for all \( x \in A \). This means that \( xa = xb \), for all \( x \in A \), which clearly implies that \( a = b \). Let \( \phi_1 \) be the mapping \( T \to a_T \) of \( M(A) \) onto \( L(H) \) given in Lemma 2.1, and let \( \phi_2 \) be the mapping \( a \to F_a \) which identifies \( L(H) \) with \( A^{**} \); \( \phi_2 \) is an isometric *-isomorphism of \( L(H) \) onto \( A^{**} \). Let \( \phi \) be the composite map \( \phi = \phi_2 \circ \phi_1 \). Then \( \phi \) is an isometric anti-isomorphism of \( M(A) \) onto \( A^{**} \) such that \( \phi(I_A) = \pi(A) \).

The following lemma is easy to prove and we state it mainly for convenience.

**Lemma 2.3.** Let \( \{A_x : \lambda \in \Lambda \} \) be a family of semisimple Banach algebras and let \( A = (\bigoplus A_x)_c \). For each \( \lambda \in \Lambda \), let \( I_\lambda = \{ f \in A : f(\mu) = 0 \text{ if } \mu \neq \lambda \} \) and \( B_\lambda = \{ f \in A : f(\lambda) = 0 \} \). Then

(i) \( I_\lambda \cap B_\lambda = (0) \) and \( I_\lambda + B_\lambda = A \).

(ii) \( l(I_\lambda) = r(I_\lambda) = B_\lambda \) and \( l(B_\lambda) = r(B_\lambda) = I_\lambda \).

**Lemma 2.4.** Let \( A, I_x, \) and \( B_x \) be as in Lemma 2.3. Let \( T \in M(A) \). Then

(i) \( T \) leaves each \( I_x \) invariant, i.e., \( T(I_x) \subseteq I_x \).

(ii) If \( T_x \) denotes the restriction of \( T \) to \( I_x \), then

\[
\| T \| = \sup_{\lambda} \| T_{I_x} \|.
\]

**Proof.** (i) Let \( x \in B_\lambda \) and \( y \in I_\lambda \). Then \( 0 = T(xy) = xTy \) which shows that \( T \in r(B_\lambda)_c I_\lambda \) by Lemma 2.3. Hence \( T(I_\lambda) \subseteq I_\lambda \).

(ii) Clearly \( \| T_{I_\lambda} \| \leq \| T \| \) for all \( \lambda \). Let \( \epsilon > 0 \) be given. Then there exists \( f \in A \), \( \| f \| = 1 \), such that \( \| T \| - \epsilon \leq \| Tf \| \). Since \( A = (\bigoplus A_x)_c \), there exists \( \lambda_1, \lambda_2, \ldots, \lambda_n \) such that \( \| f(\lambda_i) \| \leq \| T \| \) and \( \| f(\lambda) \| < \epsilon \) for \( \lambda \neq \lambda_i \) (\( i = 1, 2, \ldots, n \)). Let \( g \in A \) be such that \( g(\lambda_i) = f(\lambda_i) \) and \( g(\lambda) = 0 \) for \( \lambda \neq \lambda_i \) (\( i = 1, 2, \ldots, n \)). Then \( \| f \| = \| g \| \) and

\[
\| Tg \| = \sup_{1 \leq i \leq n} \| T_{I_x} (g(\lambda_i)) \|,
\]

so that \( \| Tg \| = \| T_{I_x} (g(\lambda_{i_0})) \| \) for some \( i_0 \), \( 1 \leq i_0 \leq n \). Since \( \| T_{I_x} (g(\lambda_{i_0})) \| \leq \| T_{I_x} \| \) for \( 1 \leq i_0 \leq n \), we have \( \| T \| - \epsilon \leq \| T_{I_x} \| \). Hence \( \| T \| = \| T \| - \epsilon \leq \| T_{I_x} \| \).

**Lemma 2.5.** Let \( \{A_x : \lambda \in \Lambda \} \) be a family of semisimple Banach algebras and let \( A = (\bigoplus A_x)_c \). Then \( M(A) \) is isometrically isomorphic to the normed full direct sum of the algebras \( M(A_x) \).
Proof. For each $\lambda \in \Lambda$, let $I_\lambda = \{ f \in A : f(\mu) = 0 \text{ if } \mu \neq \lambda \}$ and, for each $T \in M(A)$, let $T_\lambda$ be the restriction of $T$ to $I_\lambda$; $T_\lambda$ is a multiplier on $I_\lambda$. Since $A_\lambda$ is isometrically isomorphic to $I_\lambda$, each $T_\lambda$ may be identified as an element of $M(A_\lambda)$ with the same norm. For $T \in M(A)$, let $\mathcal{T}_T$ be the function on $\Lambda$ such that $\mathcal{T}_T(\lambda) = T_\lambda$. By Lemma 2.4, $\mathcal{T}_T$ is an element of the normed full direct sum $\sum M(A_\lambda)$ with $\| \mathcal{T}_T \| = \| T \|$. Hence $T \mapsto \mathcal{T}_T$ is an isometric isomorphism of $M(A)$ into $\sum M(A_\lambda)$. To show that this mapping is onto, let $\mathcal{T} \in \sum M(A_\lambda)$ and let $T$ be the mapping on $A$ such that $(Tf)(\lambda) = \mathcal{T}(\lambda)f(\lambda)$. It is easy to see that $T$ is a multiplier on $A$ with $\| T \| = \| \mathcal{T} \|$. Thus $T \mapsto \mathcal{T}_T$ is onto and this completes the proof.

Corollary 2.6. Let $A$ be a dual $B^*$-algebra and let $\{ I_\lambda : \lambda \in \Lambda \}$ be the family of all minimal closed two-sided ideals of $A$. For each $T \in M(A)$ and $\lambda \in \Lambda$, let $T_\lambda$ be the restriction of $T$ to $I_\lambda$. Let $M_\lambda = \{ T_\lambda : T \in M(A) \}$. Then $M(A)$ is isometrically isomorphic to the normed full direct sum of the algebras $M_\lambda$.

Proof. By [6, p. 267, Theorem (4.10.14)], $A = (\sum I_\lambda)_0$ and so, by Lemma 2.5, $M(A)$ is isometrically isomorphic to the normed full direct sum $\sum M(I_\lambda)$. Now, since $I_\lambda \cap r(I_\lambda) = (0)$ and $I_\lambda + r(I_\lambda) = A$, it is easy to show that $M_\lambda = M(I_\lambda)$. Hence $M(A)$ is isometrically isomorphic to $\sum M_\lambda$.

3. We are now ready to prove the characterizations mentioned in the abstract.

Theorem 3.1. Let $A$ be a $B^*$-algebra, $A^{**}$ its second conjugate space and $\pi(A)$ the canonical image of $A$ in $A^{**}$. Give $A^{**}$ the Arens product. Then $A$ is a dual algebra if and only if there exists an isometric anti-isomorphism $\phi$ of $M(A)$ onto $A^{**}$ such that $\phi(IA) = \pi(A)$.

Proof. Suppose that $A$ is dual. Then there exists a family of Hilbert spaces $\{ H_\lambda : \lambda \in \Lambda \}$ such that $A$ is $\ast$-isomorphic to $(\sum LC(H_\lambda))_0$ [4, p. 221, Lemma 2.3]. It now follows that $A^\ast$ is isometrically isomorphic to $(\sum \tau c(H_\lambda))_1$, the $L_1$-direct sum of the algebras $\tau c(H_\lambda)_1$, and that in turn $A^{**}$ is isometrically isomorphic to the normed full direct sum $\sum L(H_\lambda)$ of the algebras $L(H_\lambda)$ [8, p. 532]. Letting $LC(H_\lambda) = A_\lambda$ and identifying $A$ with $(\sum A_\lambda)_0$, Lemma 2.5 shows that $M(A)$ is isometrically isomorphic to the normed full direct sum of the algebras $M(A_\lambda)$. But, by Corollary 2.2, $M(A_\lambda)$ is isometrically anti-isomorphic to $L(H_\lambda)$, for each $\lambda \in \Lambda$. Hence $M(A)$ is isometrically anti-isomorphic to $\sum L(H_\lambda)$. Since $\sum L(H_\lambda)$ is $\ast$-isomorphic to $A^{**}$, it follows that $M(A)$ is isometrically anti-isomorphic to $A^{**}$. Let $\phi$ be this anti-isomorphism. It is now easy to deduce from Corollary 2.2 that $\phi(I_A) = \pi(A)$.

Conversely, suppose that there exists an isometric anti-isomorphism of $M(A)$ onto $A^{**}$ such that $\phi(I_A) = \pi(A)$. Since $I_A$ is a closed left ideal
of \( M(A) \), it follows that \( \pi(A) \) is a closed right ideal of \( A^{**} \). But \( \pi(A) \) is a *-subalgebra of \( A^{**} \). Hence \( \pi(A) \) is a closed two-sided ideal of \( A^{**} \). Therefore, by [8, p. 533, Theorem 5.1], \( A \) is dual. This completes the proof.

As an immediate consequence of the proof of Theorem 3.1, we have:

**Corollary 3.2.** A \( B^* \)-algebra \( A \) is dual if and only if every multiplier on \( \pi(A) \) is given by the restriction to \( \pi(A) \) of the right multiplication operator \( T_a \), for some \( a \in A^{**} \).

**Theorem 3.3.** Let \( A \) be a \( B^* \)-algebra with minimal left ideals. Let \( I \) be a minimal left ideal of \( A \), \( [I] \) the closed two-sided ideal generated by \( I \). Then \( A \) is *-isomorphic to \( L(H) \), for some Hilbert space \( H \), if and only if \( M([I]) \) is isometrically anti-isomorphic to \( A \).

**Proof.** Suppose \( A \) is *-isomorphic to \( L(H) \). Then the closed two-sided ideal generated by any minimal left ideal \( I \) of \( A \) is *-isomorphic to \( LC(H) \) and, by Corollary 2.2, \( M(LC(H)) \) is isometrically anti-isomorphic to \( L(H) \).

Conversely suppose that \( M([I]) \) is isometrically anti-isomorphic to \( A \). Let \( B \) be the closure of the socle of \( A \). Then \( B \) is a nonzero dual \( B^* \)-algebra and every minimal left ideal of \( A \) is also a minimal left ideal of \( B \). Hence, by [2, p. 158, Theorem 5], \( [I] \) is a minimal closed two-sided ideal of \( B \) and therefore is *-isomorphic to \( LC(H) \), for some Hilbert space \( H \). Hence, by Lemma 2.1, \( A \) is isometrically isomorphic to \( L(H) \). [6, p. 248, Corollary (4.8.19)] now completes the proof.

**Corollary 3.4.** Let \( A \) be a \( B^* \)-algebra containing minimal left ideals. Let \( I \) be a minimal left ideal of \( A \) and \( [I] \) the closed two-sided ideal generated by \( I \). Then \( A \) is *-isomorphic to \( L(H) \), for some Hilbert space \( H \), if and only if \( A \) is *-isomorphic to the second conjugate space of \( [I] \) considered as a \( B^* \)-algebra with Arens product.

**Proof.** This follows from the proof above and [6, p. 248, Corollary (4.8.19)].

For another characterization of the algebra \( L(H) \), see [9, p. 537, Theorem 8].

**Remark.** We observe that the Hilbert space \( H \) in Theorem 3.3 as well as in Corollary 3.4 is essentially unique. For if \( L(H_1) \) is *-isomorphic to \( L(H_2) \), then \( H_1 \) is isometrically isomorphic to \( H_2 \). (See [9, p. 538].)

**References**


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