ON THE ABSOLUTE NÖRLUND SUMMABILITY FACTORS OF INFINITE SERIES

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Abstract. A general theorem on absolute Nörlund summability factors of infinite series has been obtained. The theorem includes, as special cases, a number of well-known results. Several new results can also be deduced from it.

1. Let \( p_n \geq 0, q_n > 0, Q_n = q_0 + q_1 + \cdots + q_n \to \infty, \) \( P_n = p_0 + p_1 + \cdots + p_n \) and \( P_{-1} = p_{-1} = Q_{-1} = 0. \) Let \( \sum a_n \) be a given infinite series with \( s_n \) as its \( n \)th partial sum. The series \( \sum a_n \) is said to be summable \( |N, p_n| \) if \( t_n \in BV, \) where \( t_n = P_{n+1}^{-1} \sum_{k=0}^{n} p_{n-k} s_k, \) \( P_n \neq 0. \) It is said to be summable \( |N, q_n| \) if \( T_n \in BV, \) where \( T_n = Q_{n+1}^{-1} \sum_{k=0}^{n} q_k s_k. \)

It is well known that necessary and sufficient conditions for the method \( (N, p_n) \) to be absolutely regular are:

\[
\begin{align*}
1.1 & \quad p_n / P_n \to 0, \quad n \to \infty, \\
1.2 & \quad \sum_{n=0}^{\infty} \left| \frac{P_{n+1-k}}{P_{n+1}} - \frac{P_{n-k}}{P_n} \right| < C \quad \text{for all } k \geq 1,
\end{align*}
\]
where \( C \) is a positive constant.

2. The following theorems concerning summability factors are known.

Theorem A [1]. The necessary and sufficient conditions that \( \sum a_n e_n \) be summable \( |N, 1/n+1| \) whenever \( \sum a_n \) is summable \( |C, 1| \) are

\[
\begin{align*}
2.1 & \quad e_n = O(\log n/n), \\
2.2 & \quad \Delta e_n = O(1/n).
\end{align*}
\]

Theorem B [2]. The necessary and sufficient conditions that \( \sum a_n e_n \) be summable \( |C, \alpha|, 0 \leq \alpha \leq 1 \), whenever \( \sum a_n \) is summable \( |R, \log n, 1| \) are

\[
\begin{align*}
2.3 & \quad e_n = O(n^{\alpha-1}/\log n), \\
2.4 & \quad \Delta e_n = O(1/n (\log n)).
\end{align*}
\]

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Theorem C [3]. If a series \( \sum a_n \) is summable \( |C, 1| \) and \( \{p_n\} \) is a non-increasing sequence of real and nonnegative numbers, then the series \( \sum a_n(p_n/n) \) is summable \( |N, p_n| \).

Theorem D [4]. The necessary and sufficient conditions that \( \sum a_n\varepsilon_n \) be summable \( |C, \alpha| \), \( 0 \leq \alpha \leq 1 \), whenever \( \sum a_n \) is summable \( |C, 1| \) are

\[
\varepsilon_n = O(n^{\alpha-1}),
\]
\[
\Delta\varepsilon_n = O(n^{-1}).
\]

The object of this note is to prove a general theorem which includes, as special cases, all the results stated above. It may be remarked that the proof of our theorem is quite simple and straightforward.

Theorem. Let \((N, p_n)\) be an absolutely regular method. Let

\[
P_k \sum_{n=1}^{\infty} \left| \frac{P_{n+1-k} - P_{n-k}}{P_n} \right| < C \quad \text{for all } k \geq 0,
\]

(2.8) \( P_nq_n = O(Q_n) \).

Then the necessary and sufficient conditions for \( \sum a_n\varepsilon_n \) to be summable \( |N, p_n| \) whenever \( \sum a_n \) is summable \( |\mathcal{N}, q_n| \) are

\[
\varepsilon_n = O(Pnq_n/Q_n),
\]
\[
\Delta\varepsilon_n = O(q_n/Q_{n-1}).
\]

We deduce several other interesting results.

Corollary I. The necessary and sufficient conditions that \( \sum a_n\varepsilon_n \) be summable \( |N, 1/n+1| \) whenever \( \sum a_n \) is summable \( |R, \log n, 1| \) are

(2.11) \( \varepsilon_n = O(1/n), \)
\[
\Delta\varepsilon_n = O(1/(n \log n)).
\]

Corollary II. The necessary and sufficient conditions that \( \sum a_n\varepsilon_n \) be absolutely convergent whenever \( \sum a_n \) is summable \( |\mathcal{N}, q_n| \) are

(2.13) \( \varepsilon_n = O(q_n/Q_n), \)
\[
\Delta\varepsilon_n = O(q_n/Q_{n-1}).
\]

3. The following lemma is required for the proof of our theorem.

Lemma [5]. Let \( \{x_n\} \) be a sequence of real numbers and let its linear transformation be \( y_n = \sum_{k=0}^{\infty} a_{n,k}x_k \), where \( (a_{n,k}) \) is an infinite matrix. In order that \( \sum |x_n| < \infty \) may imply \( \sum |\Delta y_n| < \infty \), it is necessary and sufficient
that
\[ \sum_{n=0}^{\infty} |a_{n+1,m} - a_{n,m}| < C \text{ for } m = 1, 2, 3, \ldots. \]

4. Proof of theorem. Let \( T_n = Q_n^{-1} \sum_{k=0}^{n} q_k s_k \), \( x_n = T_n - T_{n-1} \) and \( t_n^* \) denote the \((N,p_n)\) mean of the series \( \sum a_n e_n \), then
\[ x_n = \frac{q_n}{Q_n Q_{n-1}} \sum_{k=1}^{n} a_k Q_{k-1} \]
and
\[ t_n^* \equiv y_n = P_n^{-1} \sum_{k=1}^{n} \frac{Q_k Q_{k-1}}{q_k} \Delta_k \left( \frac{P_{n-k} e_k}{Q_{k-1}} \right) x_k. \]
Putting
\[ b_{n,k} = (Q_k Q_{k-1}/q_k P_n) \Delta_k (P_{n-k} e_k/Q_{k-1}), \quad k \leq n, \]
\[ = 0, \quad k > n, \]
we have
\[ y_n = \sum_{k=1}^{\infty} b_{n,k} x_k. \]
 Applying the above lemma it follows that \( \sum a_n e_n \) is summable \(|N,p_n|\) whenever \( \sum a_n \) is summable \(|\bar{N},q_n|\) if and only if
\[ \sum_{n=1}^{\infty} |b_{n+1,k} - b_{n,k}| < C \text{ for } k = 1, 2, 3, \ldots. \]

Now
\[ b_{n+1,k} - b_{n,k} = \Delta e_k \frac{Q_{k-1}}{q_k} \left( \frac{P_{n-k}}{P_n} - \frac{P_{n-k-1}}{P_n} \right) + \epsilon_k Q_k \left( \frac{P_{n+1-k}}{P_n} - \frac{P_{n-k}}{P_n} \right) + \epsilon_k \frac{P_{n-k}}{P_n} - \frac{P_{n-k-1}}{P_n}. \]

Sufficiency. We have
\[ \sum_{n=1}^{\infty} |b_{n+1,k} - b_{n,k}| \leq |\Delta e_k| \frac{Q_{k-1}}{q_k} \sum_{n=1}^{\infty} \left| \frac{P_{n-k}}{P_n} - \frac{P_{n-k-1}}{P_n} \right| + CP_k \sum_{n=1}^{\infty} \left| \frac{P_{n+1-k}}{P_n} - \frac{P_{n-k}}{P_n} \right| + |\epsilon_k| \sum_{n=1}^{\infty} \left| \frac{P_{n-k}}{P_n} - \frac{P_{n-k-1}}{P_n} \right| < C \]
by the hypotheses of the theorem.

Necessity of (2.9). Since \( \sum_{n=1}^{\infty} |b_{n+1,k} - b_{n,k}| < C \) for all \( k \), it follows that the first term, namely for \( n=k-1 \), must be finite for all \( k \). Hence \( \epsilon_k = O(P_k q_k |Q_k|) \).
Necessity of (2.10). Since $e_n = O(1)$ we have

$$|\Delta e_k| \frac{Q_{k-1}}{q_k} \leq |\Delta e_k| \frac{Q_{k-1}}{q_k} \sum_{n=1}^{\infty} \left| \frac{P_{n-k}}{P_{n+1}} - \frac{P_{n-k-1}}{P_n} \right|$$

$$\leq \sum_{n=1}^{\infty} |b_{n+1,k} - b_{n,k}|$$

$$+ |e_k| \frac{Q_k}{q_k} \sum_{n=1}^{\infty} \left| \frac{P_{n+1-k}}{P_{n+1}} - \frac{P_{n-k}}{P_n} \right| + \sum_{n=1}^{\infty} \left| \frac{P_{n-k}}{P_{n+1}} - \frac{P_{n-k-1}}{P_n} \right|$$

$$< C$$

for all $k$. Thus $\Delta e_k = O(q_k/Q_{k-1})$.

This completes the proof of the theorem.

REFERENCES


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