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ON THE ABSOLUTE NÖRLUND SUMMABILITY FACTORS OF INFINITE SERIES

S. M. MAZHAR

ABSTRACT. A general theorem on absolute Nörlund summability factors of infinite series has been obtained. The theorem includes, as special cases, a number of well-known results. Several new results can also be deduced from it.

1. Let $p_n \geqq 0$, $q_n > 0$, $Q_n = q_0 + q_1 + \cdots + q_n \to \infty$, $P_n = p_0 + p_1 + \cdots + p_n$ and $P_{-1} = p_{-1} = Q_{-1} = 0$. Let $\sum a_n$ be a given infinite series with s_n as its nth partial sum. The series $\sum a_n$ is said to be summable $|N, p_n|$ if $t_n \in BV$, where $t_n = P_n^{-1} \sum_{k=0}^{n} p_{n-k} s_k$, $P_n \neq 0$. It is said to be summable $|\bar{N}, q_n|$ if $T_n \in BV$, where $T_n = Q_n^{-1} \sum_{k=0}^{n} q_k s_k$.

It is well known that necessary and sufficient conditions for the method (N, p_n) to be absolutely regular are:

$$p_n/P_n \to 0, \qquad n \to \infty, \tag{1.1}$$

$$\sum_{n=0}^{\infty} \left| \frac{P_{n+1-k}}{P_{n+1}} - \frac{P_{n-k}}{P_n} \right| < C \quad \text{for all } k \geqq 1, \tag{1.2}$$

where C is a positive constant.

2. The following theorems concerning summability factors are known.

THEOREM A [1]. *The necessary and sufficient conditions that $\sum a_n \varepsilon_n$ be summable $|N, 1/n+1|$ whenever $\sum a_n$ is summable $|C, 1|$* are

$$\varepsilon_n = O(\log n/n), \tag{2.1}$$

$$\Delta \varepsilon_n = O(1/n). \tag{2.2}$$

THEOREM B [2]. *The necessary and sufficient conditions that $\sum a_n \varepsilon_n$ be summable $|C, \alpha|$, $0 \leqq \alpha \leqq 1$, whenever $\sum a_n$ is summable $|R, \log n, 1|$ are*

$$\varepsilon_n = O(n^{\alpha-1}/\log n), \tag{2.3}$$

$$\Delta \varepsilon_n = O(1/n\,(\log n)). \tag{2.4}$$

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THEOREM C [3]. *If a series $\sum a_n$ is summable $|C, 1|$ and $\{p_n\}$ is a non-increasing sequence of real and nonnegative numbers, then the series $\sum a_n(P_n/n)$ is summable $|N, p_n|$.*

THEOREM D [4]. *The necessary and sufficient conditions that $\sum a_n\varepsilon_n$ be summable $|C, \alpha|$, $0 \leqq \alpha \leqq 1$, whenever $\sum a_n$ is summable $|C, 1|$ are*

$$\varepsilon_n = O(n^{\alpha-1}), \tag{2.5}$$

$$\Delta\varepsilon_n = O(n^{-1}). \tag{2.6}$$

The object of this note is to prove a general theorem which includes, as special cases, all the results stated above. It may be remarked that the proof of our theorem is quite simple and straightforward.

THEOREM. *Let (N, p_n) be an absolutely regular method. Let*

$$P_k \sum_{n=1}^{\infty} \left| \frac{p_{n+1-k}}{P_{n+1}} - \frac{p_{n-k}}{P_n} \right| < C \quad \textit{for all } k \geqq 0, \tag{2.7}$$

$$P_n q_n = O(Q_n). \tag{2.8}$$

Then the necessary and sufficient conditions for $\sum a_n\varepsilon_n$ to be summable $|N, p_n|$ whenever $\sum a_n$ is summable $|\bar{N}, q_n|$ are

$$\varepsilon_n = O(P_n q_n/Q_n), \tag{2.9}$$

$$\Delta\varepsilon_n = O(q_n/Q_{n-1}). \tag{2.10}$$

We deduce several other interesting results.

COROLLARY I. *The necessary and sufficient conditions that $\sum a_n\varepsilon_n$ be summable $|N, 1/n+1|$ whenever $\sum a_n$ is summable $|R, \log n, 1|$ are*

$$\varepsilon_n = O(1/n), \tag{2.11}$$

$$\Delta\varepsilon_n = O(1/(n \log n)). \tag{2.12}$$

COROLLARY II. *The necessary and sufficient conditions that $\sum a_n\varepsilon_n$ be absolutely convergent whenever $\sum a_n$ is summable $|\bar{N}, q_n|$ are*

$$\varepsilon_n = O(q_n/Q_n), \tag{2.13}$$

$$\Delta\varepsilon_n = O(q_n/Q_{n-1}). \tag{2.14}$$

3. The following lemma is required for the proof of our theorem.

LEMMA [5]. *Let $\{x_n\}$ be a sequence of real numbers and let its linear transformation be $y_n = \sum_{k=0}^{\infty} a_{n,k}x_k$, where $(a_{n,k})$ is an infinite matrix. In order that $\sum |x_n| < \infty$ may imply $\sum |\Delta y_n| < \infty$, it is necessary and sufficient* *that*

$$\sum_{n=0}^{\infty} |a_{n+1,m} - a_{n,m}| < C \quad \textit{for } m = 1, 2, 3, \cdots.$$

4. **Proof of theorem.** Let $T_n = Q_n^{-1} \sum_{k=0}^{n} q_k s_k$, $x_n = T_n - T_{n-1}$ and t_n^* denote the (N, p_n) mean of the series $\sum a_n \varepsilon_n$, then

$$x_n = \frac{q_n}{Q_n Q_{n-1}} \sum_{k=1}^{n} a_k Q_{k-1}$$

and

$$t_n^* \equiv y_n = P_n^{-1} \sum_{k=1}^{n} \frac{Q_k Q_{k-1}}{q_k} \Delta_k \left(\frac{P_{n-k} \varepsilon_k}{Q_{k-1}} \right) x_k.$$

Putting

$$\begin{aligned} b_{n,k} &= (Q_k Q_{k-1}/q_k P_n) \Delta_k (P_{n-k} \varepsilon_k / Q_{k-1}), & k &\leqq n, \\ &= 0, & k &> n, \end{aligned}$$

we have

$$y_n = \sum_{k=1}^{\infty} b_{n,k} x_k.$$

Applying the above lemma it follows that $\sum a_n \varepsilon_n$ is summable $|N, p_n|$ whenever $\sum a_n$ is summable $|\bar{N}, q_n|$ if and only if

$$\sum_{n=1}^{\infty} |b_{n+1,k} - b_{n,k}| < C \quad \text{for } k = 1, 2, 3, \cdots.$$

Now

$$\begin{aligned} b_{n+1,k} - b_{n,k} = \Delta \varepsilon_k \frac{Q_{k-1}}{q_k} \left(\frac{P_{n-k}}{P_{n+1}} - \frac{P_{n-k-1}}{P_n} \right) + \frac{\varepsilon_k}{q_k} Q_k \left(\frac{p_{n+1-k}}{P_{n+1}} - \frac{p_{n-k}}{P_n} \right) \\ + \varepsilon_k \left(\frac{P_{n-k}}{P_{n+1}} - \frac{P_{n-k-1}}{P_n} \right). \end{aligned}$$

Sufficiency. We have

$$\begin{aligned} \sum_{n=1}^{\infty} |b_{n+1,k} - b_{n,k}| &\leqq |\Delta \varepsilon_k| \frac{Q_{k-1}}{q_k} \sum_{n=1}^{\infty} \left| \frac{P_{n-k}}{P_{n+1}} - \frac{P_{n-k-1}}{P_n} \right| \\ &\quad + C P_k \sum_{n=1}^{\infty} \left| \frac{p_{n+1-k}}{P_{n+1}} - \frac{p_{n-k}}{P_n} \right| + |\varepsilon_k| \sum_{n=1}^{\infty} \left| \frac{P_{n-k}}{P_{n+1}} - \frac{P_{n-k-1}}{P_n} \right| \\ &< C \end{aligned}$$

by the hypotheses of the theorem.

Necessity of (2.9). Since $\sum_{n=1}^{\infty} |b_{n+1,k} - b_{n,k}| < C$ for all k, it follows that the first term, namely for $n = k-1$, must be finite for all k. Hence $\varepsilon_k = O(P_k q_k / Q_k)$.

Necessity of (2.10). Since $\varepsilon_n = O(1)$ we have

$$
\begin{aligned}
|\Delta\varepsilon_k| \frac{Q_{k-1}}{q_k} &\leqq |\Delta_{\varepsilon k}| \frac{Q_{k-1}}{q_k} \sum_{n=1}^{\infty} \left| \frac{P_{n-k}}{P_{n+1}} - \frac{P_{n-k-1}}{P_n} \right| \\
&\leqq \sum_{n=1}^{\infty} |b_{n+1,k} - b_{n,k}| \\
&\quad + \frac{|\varepsilon_k|\, Q_k}{q_k} \sum_{n=1}^{\infty} \left| \frac{p_{n+1-k}}{P_{n+1}} - \frac{p_{n-k}}{P_n} \right| + |\varepsilon_k| \sum_{n=1}^{\infty} \left| \frac{P_{n-k}}{P_{n+1}} - \frac{P_{n-k-1}}{P_n} \right| \\
&< C
\end{aligned}
$$

for all k. Thus $\Delta\varepsilon_k = O(q_k/Q_{k-1})$.

This completes the proof of the theorem.

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Department of Mathematics, Faculty of Engineering, Aligarh Muslim University, Aligarh, India