CONVOLUTION OF \(L(p, q)\) FUNCTIONS

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Abstract. In the present paper, examples are given to show that the convolution theorem, which is the \(L(p, q)\) analogue of Young's inequality for the \(L^p\) spaces, is best possible. This result is then used to obtain a theorem about bounded linear translation invariant operators between two \(L(p, q)\) spaces.

Introduction. Let \(G\) be a noncompact, locally compact group and let \(L^p = L^p(G)\), \(1 \leq p \leq \infty\), denote the Lebesgue spaces associated with \(G\). N. W. Rickert [5] has proved that the convolution of two \(L^p\) functions need not exist. In this paper we extend this result to the \(L(p, q)(G)\) spaces. For our purposes this is done first where the underlying group is one of the Euclidean \(n\) spaces \(R^n\), \(R^1 = R\) and the convolution of two measurable functions \(f\) and \(g\) defined on \(R^n\) is ordinary convolution. That is,

\[
    f * g(y) = \int_{R^n} f(y - x)g(x) \, dx, \quad y \in R^n.
\]

This result is then used to obtain a theorem about bounded linear translation invariant operators between two \(L(p, q)\) spaces.

For a discussion of the \(L(p, q)\) spaces the reader is referred to R. A. Hunt [4].

1. Let \(R^n, R^1 = R\) be the Euclidean \(n\) spaces and let \(L(p, q) = L(p, q)(R^n)\) be a Lorentz space for which \(f \in L(p, q)\), implies \(f\) is locally integrable. Namely, we will consider only those \(L(p, q)\) spaces for the particular choice of indices \(1 < p < \infty, 0 < q \leq \infty; p = q = 1\) or \(p = q = \infty\). (The spaces \(L(\infty, q), 0 < q < \infty\), will be omitted since \(f \in L(\infty, q)\), implies \(f = 0\) a.e.)

Theorem 1. If \(1 < p_i \leq \infty, 0 < q_i \leq \infty (i = 1, 2)\), where

(1) \(1/p_1 + 1/p_2 < 1\) or

(2) \(1/p_1 + 1/p_2 = 1, 1/q_1 + 1/q_2 < 1,\)

then there are functions \(f \in L(p_1, q_1), \ g \in L(p_2, q_2)\) such that \(f * g(y) = \infty\) for \(y \in R^n\).

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Proof. Assume \( n=1 \). We first prove (1). If \( p_1<\infty \) or \( p_2<\infty \), then choose \( \alpha>0 \) such that \( \alpha(1/p_1+1/p_2)=1 \) and define
\[
f(x) = \begin{cases} 
1, & \text{if } |x| \leq 1, \\
|x|^{-\alpha/p_1}, & \text{if } |x| > 1.
\end{cases}
\]
Define \( g(x) \) similarly, only this time with respect to \( \alpha \) and \( p_2 \). In the case \( p_1=p_2=\infty \), define \( f(x)=g(x)=1 \) for \( x \) in \( \mathbb{R} \).

It is easy to show that the rearrangement functions \( f^* \) and \( g^* \) of the functions \( f \) and \( g \) are given by
\[
f^*(t) = f(t/2), \quad t > 0, \\
g^*(t) = g(t/2), \quad t > 0.
\]
Then a calculation of \( \|f^*\|_{(p_1, q_1)} \) and \( \|g^*\|_{(p_2, q_2)} \), which is straightforward, shows that \( f \in L(p_1, q_1) \) and \( g \in L(p_2, q_2) \).

We next show that \( f^*g(y)=\infty \) for each \( y \) in \( \mathbb{R} \). By symmetry it is enough to assume \( y \geq 0 \).
\[
f^*g(y) = \int_{-\infty}^{\infty} f(y-x)g(x) \, dx \\
\geq \int_{1+y}^{\infty} (x-y)^{-2/p_1}x^{-2/p_2} \, dx \geq \int_{1+y}^{\infty} x^{-1} \, dx = \infty.
\]
To prove (2), choose \( \alpha, \beta \) such that \( 0<\alpha<q_1, 0<\beta<q_2, 0\leq 1/(q_1-\alpha)+1/(q_2-\beta)<1 \) and define
\[
f(x) = \begin{cases} 
2^{-1/p_1}(\log 2)^{1/(q_1-\alpha)}, & \text{if } |x| \leq 2, \\
|x|^{-1/p_1} (\log |x|)^{1/(q_2-\beta)}, & \text{if } |x| > 2.
\end{cases}
\]
Define \( g(x) \) similarly, only this time with respect to \( \beta, p_2, q_2 \) and proceed as in the proof of part (1).

Finally, assume \( n>1 \). Let \( h \) defined on \( \mathbb{R}^{n-1} \) be a function of rapid descent (see [3, p. 97]) such that \( h(y)>0 \), for \( y \) in \( \mathbb{R}^{n-1} \). If the indices \( p_1, p_2, q_1, q_2 \) are as in part (1) or (2), put \( f_1(x_1)=f(x_1) \), where \( f \) is as before and define \( f(x)=f_1(x_1)h(x_2, \ldots, x_n), \ x \in \mathbb{R}^n \). Define \( g(x) \) similarly. The rest of the proof now becomes straightforward. We omit the details.

We remark that part (1) of Theorem 1 is an adaptation of a theorem of N. W. Rickert [5] about \( L^p \), \( 1<p \leq \infty \), spaces. Based on Rickert's theorem for the \( L^p \) spaces, Theorem 1 has the following generalization: If the indices \( p_1, p_2, q_1, q_2 \) are as in Theorem 1 and \( G \) is a noncompact, locally compact group, then there is an open set \( U \) in \( G \) and there are functions \( f \in L(p_1, q_1)(G), \ g \in L(p_2, q_2)(G) \) such that \( f^*g(y) \) is not defined for \( y \) in \( U \).

Remark. Theorem 1 has a converse. It can be shown that (see [1]): If \( 1 \leq p_i \leq \infty, 0<q_i \leq \infty \) (\( i=1, 2 \), where (1) \( 1/p_1+1/p_2>1 \) or (2) \( 1/p_1+1/p_2=1, 1/q_1+1/q_2\geq 1 \), then \( f \in L(p_1, q_1) \) and \( g \in L(p_2, q_2) \) implies \( f^*g \in L(r,s) \).
where $1/r = 1/p_1 + 1/p_2 - 1$, $1/s = 1/q_1 + 1/q_2$. Moreover,

$$\|f * g\|_{r,s} \leq B \|f\|_{p_1,q_1} \|g\|_{p_2,q_2},$$

where $B > 0$ is a constant which depends only on the indices $r$, $s$, $p_i$, $q_i$ $(i = 1, 2)$.

2. Let $L(p,q) = L(p,q)(R^n)$, where $R^n$, $R^l = R$ is one of the Euclidean $n$ spaces. If $h \in R^n$, denote by $\tau_h$ the operator defined by $(\tau_h f)(x) = f(x+h)$, $x \in R^n$, for each measurable function $f$ defined on $R^n$. A bounded linear operator $T$ from $L(p_1,q_1)$ to $L(p_2,q_2)$ is translation invariant if $\tau_h T = T \tau_h$, $h \in R^n$. Such operators between the classical $L^p$, $1 \leq p \leq \infty$, spaces were studied by L. Hörmander [3].

Let $L_0(p,q)$ be the subspace of $L(p,q)$ obtained by taking the closure in $L(p,q)$, with respect to the metric topology of $L(p,q)$, of the set of simple functions having compact support. It can be shown that (see [1]):

1. If $1 < p < \infty$, $0 < q < \infty$ or $p = q = 1$, then $L_0(p,q) = L(p,q)$; (2) If $p = q = \infty$, then $L_0(\infty, \infty) = L_0^\infty$ is the subspace of functions in $L^\infty$ which tend to 0 at infinity. (3) If $1 < p < \infty$, $q = \infty$, then $L_0(p,\infty)$ is the subspace of $L(p,\infty)$ consisting of those functions for which $t^{1/p} f^*(t)$ converges to 0 as $t$ tends to $0^+$ and $\infty$.

L. Hörmander [3, p. 96] showed that nontrivial translation-invariant operators need not exist between two $L^p$ spaces. Using the definition of a rearrangement function $f^*$ of a function $f$ and Hörmander’s proof, this result extends easily to the $L(p,q)$ spaces. And takes the form: If $T$ is a bounded linear translation invariant operator from $L(p_1,q_1)$ to $L_0(p_2,q_2)$, such that $p_1 > p_2$, then $T = 0$ when restricted to $L_0(p_1,q_1)$.

Our purpose here is to prove the following theorem.

**Theorem 2.** If $T$ is a bounded linear translation invariant operator from $L(p,q_1)$ to $L(p,q_2)$, $1 < p < \infty$, such that

1. $f \geq 0$, implies $Tf \geq 0$,
2. $q_1 > q_2$,

then $T = 0$ if $q_1 < \infty$ and if $q_1 = \infty$ then the restriction of $T$ to $L_0(p,\infty)$ is 0.

**Proof.** If $T$ is a bounded linear translation invariant operator from $L(p_1,q_1)$ to $L(p_2,q_2)$, then $T$ has the characterization (see [1]) that: if $g \in L_1(1,1)$ and $f \in L_0(p_1,q_1)$, then $[T(f * g)](y) = [g * Tf](y)$, $y \in R^n$. This will be used later in the proof.

The continuity of $T$ implies the existence of a constant $B \geq 0$ such that

$$\|Tf\|_{p_2,q_2} \leq B \|f\|_{p_1,q_1}, \quad f \in L(p,q_1).$$

If $B = 0$, then $T = 0$. Assume $B > 0$ and choose indices $r$, $s$ such that $1/p + 1/r = 1$, $1/s + 1/q_2 = 1$. Since $q_1 > q_2$, it follows that $1/s + 1/q_1 < 1$. Let
Let $f \in L(p, q_1)$ and $g \in L(r, s)$ be as in Theorem 1. Let

$$E_n = \{x: 1/n < \langle f(x) \rangle \leq n\} \cap \{x: |x| \leq n\},$$

$$F_n = \{x: 1/n < f(x) \leq n\} \cap \{x: |x| \leq n\}, \quad n = 1, 2, \cdots.$$ 

Put $f_n(x) = f(x)\chi_{E_n}(x)$ and $g_n(x) = g(x)\chi_{F_n}(x)$, where $\chi_E$ denotes the characteristic function of the measurable set $E$. It follows that $f_n \in L(1, 1) \cap L_0(p, q_1)$ and $g_n \in L(1, 1) \cap L_0(r, s)$, $n = 1, 2, \cdots$. Moreover, $f_n \uparrow f$ and $g_n \downarrow g$ pointwise everywhere as $n$ tends to infinity.

Next, let $u(x)$ defined on $\mathbb{R}^n$ be a nonnegative simple function having compact support. Using the remarks at the end of §1 we have

$$B \|u\|_{(1, 1)} \|f\|_{(p, q_1)} \|g\|_{(r, s)} \geq B \|u\|_{(1, 1)} \|f_n\|_{(p, q_1)} \|g_n\|_{(r, s)} \geq \|u\|_{(1, 1)} \|Tf_n\|_{(p, q_2)} \|g_n\|_{(r, s)} \geq C \|u \ast Tf_n \ast g_n\|_{(\infty, \infty)} \geq C \|u \ast g_n \|_{(\infty, \infty)} \geq C(f \ast g \ast Tu)(y),$$

$y \in \mathbb{R}^n$, and a constant $C > 0$ which depends only on the indices $p, q_1, r, s, 1, q_1$. By condition (1) and the monotone convergence theorem, it follows that

$$B \|u\|_{(1, 1)} \|f\|_{(p, q_1)} \|g\|_{(r, s)} \geq C(f \ast g \ast Tu)(y), \quad y \in \mathbb{R}^n.$$

By the way the functions $f$ and $g$ were constructed, the term on the right equals $\infty$ for each $y$ in $\mathbb{R}^n$, unless $Tu(x) = 0$ a.e. $x \in \mathbb{R}^n$.

If $u$ is an arbitrary simple function having compact support, then $u = u^+ - u^-$, where $u^+$ and $u^-$ are nonnegative simple functions having compact support. By the linearity of the operator $T$ we have $Tu = Tu^+ - Tu^- = 0$. This implies $T = 0$ on $L_0(p, q_1)$.

Remark. If for the choice of indices $p_1, p_2, q_1, q_2$ either $p_1 < p_2$ or $p_1 = p_2, q_1 \leq q_2$, then by using the remark at the end of §1 it is easy to construct a nontrivial bounded linear translation invariant operator from $L(p_1, q_1)$ to $L(p_2, q_2)$.

**REFERENCES**


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