JACOBI'S BOUND FOR FIRST ORDER DIFFERENCE EQUATIONS

BARBARA A. LANDO

Abstract. Let $A_1, \ldots, A_n$ be a system of difference polynomials in $y^{(1)}, \ldots, y^{(n)}$, and let $\mathcal{M}$ be an irreducible component of the difference variety $\mathcal{M}(A_1, \ldots, A_n)$. If $r_{ij}$ is the order of $A_i$ in $y^{(j)}$, the Jacobi number $J$ of the system is defined to be
$$\max_{\sigma \in S_n} \sum_{j=1}^{n} r_{ij} \sigma$$
where $\sigma$ is a permutation of $1, \ldots, n$. In this paper it is shown for first order systems that if $\dim \mathcal{M} = 0$, then $\text{Ord} \mathcal{M} \leq J$. The methods used are analogous to those used to obtain the corresponding result for differential equations (given in a recent paper by the author).

1. Introduction. For any system of difference polynomials $A_1, \ldots, A_n$, with $r_{ij}$ the order of $A_i$ in $y^{(j)}$, the Ritt number
$$R = \sum_{j=1}^{n} \max_{i} \{r_{ij} : i = 1, \ldots, n\}$$
provides a bound for the effective order of an irreducible component $\mathcal{M}$ of dimension 0 [1, pp. 253–255]. Greenspan improved this bound for those systems in which every irreducible component has dimension 0. (See [1, pp. 256–258].) In the case $n=2$, the Greenspan number coincides with the Jacobi number $J$. The Jacobi number can also be verified for linear difference systems by a proof analogous to that for the differential case [3].

The definitions of [1] are assumed, and the notation of [2] will be used.

2. Specialization problem. Let $K$ be an inversive difference field with automorphism $\tau$. Let $\bar{R}$ be a difference kernel with principal realization $\bar{\sigma}$. In general, if $R$ is a kernel which specializes to $\bar{R}$, there may be no principal realization of $R$ which specializes to $\bar{\sigma}$. (See [1, p. 322, Example 2].) The following theorem presents in one case which such a specialization of principal realizations does exist.

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Theorem 1. Let $K$ be inversive. Let $R$ and $\tilde{R}$ be difference kernels consisting of $K(a, \cdots, a_r)$, $T_r$, and $K(\tilde{a}, \cdots, \tilde{a}_r)$, $\tilde{T}_r$, respectively, $r \geq 0$, with $K(a, \cdots, a_{r-1}) \cong_K K(\tilde{a}, \cdots, \tilde{a}_{r-1})$. Let $\tilde{a}$ be a principal realization of $\tilde{R}$. If $R$ specializes to $\tilde{R}$, then there exists a principal realization $\alpha$ of $R$ which specializes to $\tilde{a}$.

Proof. It may be assumed that the kernels are of length 0 or 1 [1, p. 160]. The proof is given for kernels of length 1 (the length 0 case is similar). By an argument like that in Theorem 1 of [2], we may assume that $a=\tilde{a}$, $\deg R=0$ and $\deg R=1$.

It will be sufficient to show that for any $h>1$ a kernel $K(a, \cdots, a_h)$, $T_h$, can be found with $K(a, \cdots, a_{h-1})$ a generic prolongation of $K(a, \cdots, a_h)$, $k=1, \cdots, h-1$, such that $(a, \cdots, a_k)$ specializes to $(\tilde{a}, \tilde{a}_2, \cdots, \tilde{a}_h)$, where $\tilde{a}$ is the principal realization of $\tilde{R}$. For suppose the theorem is false: then no principal realization of $R$ specializes to $\tilde{a}$. Since the number of distinct principal realizations is finite [1, p. 156], there exists an integer $H>1$ such that for any principal realization $\alpha$ of $R$, $(a, \cdots, a_H)$ does not specialize to $(\tilde{a}, \cdots, \tilde{a}_H)$.

Let $L$ be the algebraic closure of $K(\tilde{a})$. By taking free joins and isomorphic kernels if necessary, we may assume that $(\tilde{a}, \tilde{a}_1)=(a, a_1)$, that $t.d. L(a_1)/L=1$, and that $(a, a_1)\sim_T (a, a_1)$. Then for any given $h>1$, there exist vectors $b_2, \cdots, b_h$ such that

(i) $K(a, b_2, \cdots, b_h) \cong_K K(\tilde{a}, \tilde{a}_2, \cdots, \tilde{a}_h)$,
(ii) $(a, a_1, b_2, \cdots, b_h) \sim_T (a, a_1, \tilde{z}_2, \cdots, \tilde{z}_h)$,
(iii) $t.d. L(a, a_1, b_2, \cdots, b_h)/L=1$

(by Lemma 1 of [2] and the isomorphism $K(a_1) \cong_K K(\tilde{a}_1)$).

Since (ii) and (iii) hold, Lemma 2 of [2] may be applied to obtain parameter $t \in L(a_1, b_2, \cdots, b_h)$, transcendental over $L$, such that $L[a_1, b_2, \cdots, b_h] \subseteq L[[t]]$ with

$$a_1^{(i)} = a_1^{(i)} + \sum_{j=1}^{\infty} c_{ij}t^j, \quad i = 1, \cdots, n;$$

$$b_k^{(i)} = \tilde{b}_k^{(i)} + \sum_{j=1}^{\infty} d_{kij}t^j, \quad i = 1, \cdots, n; \quad k = 2, \cdots, h.$$

The transformation of the kernel $R$, $T: K(a) \to K(a_1)$, may be extended to an isomorphism $T$ of $K(a, \tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_{h-1})$ onto $K(a_1, b_2, \cdots, b_h)$, obtained by the composition of $T$ and the isomorphism of (i):

$$T: K(a, \tilde{a}_1, \cdots, \tilde{a}_{h-1}) \xrightarrow{T} K(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_h) \xrightarrow{K} K(a_1, b_2, \cdots, b_h).$$
T may then be extended to a monomorphism of $L$ into a field $L_1$, which is the algebraic closure of the quotient field of $L[[t]]$. Then $T$ extends to a monomorphism of $L[[t]]$ into $L_1[[t_1]]$, where $t_1$ is transcendental over $L_1$ and $T(t)=t_1$. Repeating this method of extension, for each $k=2, \cdots, h-1$, one may extend $T$ to a monomorphism $T: L_{k-1}[[t_{k-1}]] \rightarrow L_k[[t_k]]$, with $L_{k-1}[[t_{k-1}]] \subseteq L_k$ and $t_k$ transcendental over $L_k$.

Since $a_i \in L[[t]]$, $T^{k\alpha}=T^{k-1}a_i \in L_{k-1}[[t_{k-1}]]$, $k=2, \cdots, h$. Let $a_k=T^k\alpha$. Then $K[a, a_1, \cdots, a_{h-1}] \subseteq L_{h-2}[[t_{h-2}]]$, and $T$ restricted to $K[a, a_1, \cdots, a_{h}]$ is an isomorphism onto $K[a_1, \cdots, a_{h}]$. Thus $K(a, a_1, \cdots, a_{h})$ is a kernel which can be obtained from $R$ by $h-1$ prolongations.

In the following let $b$ denote the vector $(b_2, \cdots, b_h)$. It will be shown that $K(a, a_1, \cdots, a_{h})$ is obtained by generic prolongations. Since $t_e L(a_1, b)$, $t_k=T^k t_e L_k(a_{k+1}, T^k b)$, $1 \leq k \leq h-1$. But $t_k$ is transcendental over $L_k$; thus

$$\text{(2)} \quad \text{t.d.} L_k(a_{k+1}, T^k b)/L_k \geq 1.$$

It may be noted that

$$\text{t.d.} K(a, \cdots, a_{k+1}, T^k b)/K(a, \cdots, a_{k+1}) \leq \text{t.d.} K(a_{k+1}, T^k b)/K(a_{k+1}) = \text{t.d.} K(a_1, b)/K(a_1) \geq \text{t.d.} L_k(a_{k+1}, T^k b)/L_k \geq 1.$$

with the last 2 equalities following from (i) and the fact that $\deg R=0$, respectively.

Using (2) and (3), one obtains

$$\text{t.d.} K(a, \cdots, a_{k+1})/K(a, \cdots, a_{k+1}) \leq \text{t.d.} K(a_{k+1}, T^k b)/K(a_{k+1}) = \text{t.d.} K(a_1, \cdots, a_{h})/K(a_1) \geq \text{t.d.} L_k(a_{k+1}, T^k b)/L_k \geq 1.$$

However,

$$\text{t.d.} K(a, \cdots, a_{k+1})/K(a, \cdots, a_{k+1}) \leq \text{t.d.} K(a_{k}, a_{k+1})/K(a_k) = \text{t.d.} K(a, a_1)/K(a) = \deg R = 1.$$

Hence, $\text{t.d.} K(a, \cdots, a_{k+1})/K(a, \cdots, a_{k+1})=1$ for each $k$. Thus $K(a, \cdots, a_{h})$ is obtained from $R$ by generic prolongations.

Let $\Phi_k: L_k[[t_k]] \rightarrow L_k$ be the homomorphism over $L_k$ defined by $\Phi_k t_k=0$, $0 \leq k \leq h-1$. From the series expansions (1), we note that

$$a_2^{(i)} = T a_1^{(i)} = T a_1^{(i)} + \sum_{j=1}^{\infty} (T c_{ij})t_1^j = b_2^{(i)} + \sum_{j=1}^{\infty} (T c_{ij})t_1^j = \bar{a}_2^{(i)} + \sum_{j=1}^{\infty} d_{2ij}t^j + \sum_{j=1}^{\infty} (T c_{ij})t_1^j.$$

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and thus

\[ a_k^{(i)} = \bar{a}_k^{(i)} + \sum_{j=1}^{\infty} d_{ki} t^j + \sum_{j=1}^{\infty} (T d_{k-1i}) t^j + \cdots + \sum_{j=1}^{\infty} (T^{k-1} c_{i}) t^j, \]

for \( k \geq 2 \). \( K[a, \cdots, a_{k+1}] \subseteq L_k[[t_k]] \). For \( q \leq k \), \( a_q \in L_{k-1}[[t_{k-1}]] \) and \( \Phi_k a_q = a_q \). But

\[ \Phi_k a_k^{(i)} = \Phi_k \left( \bar{a}_k^{(i)} + \sum_{j=1}^{\infty} d_{k+1i} t^j + \cdots + \sum_{j=1}^{\infty} (T^{k-1} c_{i}) t^j \right) \]

\[ = \bar{a}_k^{(i)} + \cdots + \sum_{j=1}^{\infty} (T^{k-1} d_{k+1i}) t^j \in L_{k-1}[[t_{k-1}]]. \]

Hence \( \Phi = \Phi_0 \Phi_1 \cdots \Phi_{k-1} \) is a well-defined homomorphism on \( K[a, \cdots, a_h] \) with \( \Phi a_k = \bar{a}_k \), \( k = 0, \cdots, h \). Thus

\[ (a, \cdots, a_h) \rightarrow (a, \bar{a}_1, \cdots, \bar{a}_h). \]

**Corollary.** Every regular realization of a kernel is the specialization of a principal realization.

**Proof.** By the same argument as in Corollary to Theorem 1 of [2].

(This result has been obtained by other methods [1, p. 189].)

**3. Jacobi's bound.** Let \( K\{y\} = K\{y^{(1)}, \cdots, y^{(n)}\} \) be a polynomial difference ring over a difference field \( K \). Let \( A_1, \cdots, A_m \) be a system of difference polynomials in \( K\{y\} \). Let \( r_{ij} \) be the order of \( A_i \) in \( y^{(j)} \), with \( r_{ij} = 0 \) if \( y^{(j)} \) does not effectively appear in \( A_i \). The Jacobi number of the system, \( J(A) \), is the maximal diagonal sum of the matrix \( |r_{ij}| \) (see [2]).

**Theorem 2.** Let \( A_1, \cdots, A_m \) be first order difference polynomials in \( K\{y\} \). If \( \mathcal{M} \) is an irreducible component of \( \mathcal{M}(A_1, \cdots, A_m) \) of dimension 0, then \( E \) ord \( \mathcal{M} \leq J(A) \).

**Proof.** We may assume that \( K \) is inversive and show that ord \( \mathcal{M} \leq J(A) \). (See the proof of Theorem IX in [1, Chapter 8].) Let \( \bar{a} \) be a generic zero of \( \mathcal{M} \). Then \( \bar{a} \) is a principal realization of the kernel \( \bar{R} \) with field \( K(\bar{a}, \bar{a}_1) \) (argument as in Theorem 3 of [2]). \( \deg \bar{R} = \dim \mathcal{M} = 0 \), and ord \( \mathcal{M} = \text{t.d.} \ K(\bar{a})/K \).

\[ \text{Since } (\bar{a}, \bar{a}_1) \text{ is a zero of the ideal } (A_1, \cdots, A_m) \text{ in } K[y, y_1], \text{ there is a generic zero of some component of } \mathcal{M}(A_1, \cdots, A_m) \text{ such that} \]

\[ (a, a_1) \rightarrow (\bar{a}, \bar{a}_1). \]

Thus t.d. \( K(a)/K = \text{t.d.} K(\bar{a})/K = s \), and t.d. \( K(a_1)/K = \text{t.d.} K(\bar{a}_1)/K = r \), for some \( s \geq 0 \) and \( r \geq 0 \).
By two applications of Lemma 4 of [2], one obtains \( c \) and \( \alpha_1 \) such that

\[
\begin{align*}
(a, a_1) &\xrightarrow{K} (c, \tilde{a}_1) \xrightarrow{K} (\tilde{a}, \alpha_1) \xrightarrow{K} (\tilde{a}, \tilde{a}_1)
\end{align*}
\]

with \( \text{t.d. } K(a, a_1)/K - \text{t.d. } K(c, \tilde{a}_1)/K \leq r \) and

\[
\text{t.d. } K(c, \tilde{a}_1)/K - \text{t.d. } K(\tilde{a}, \alpha_1)/K \leq s.
\]

Therefore,

\[
\text{t.d. } K(a, a_1)/K - \text{t.d. } K(\tilde{a}, \alpha_1)/K
\leq \text{t.d. } K(a_1)/K - \text{t.d. } K(\tilde{a}_1)/K + \text{t.d. } K(a)/K - \text{t.d. } K(\tilde{a})/K.
\]

Since \((\tilde{a}_1)_{K'(\tilde{a}_1)}(\alpha_1)_{K'(\alpha_1)}, K(\alpha_1) \cong K(\tilde{a}_1)\). Thus, since \(K(\tilde{a}, \tilde{a}_1)\) is a kernel, \(K(\tilde{a}, \alpha_1)\) is a kernel \(R\). By Theorem 1, there is a principal realization \(\alpha\) of \(R\) which specializes to \(\tilde{a}\). But \(\alpha\) is a zero of \(A_1, \ldots, A_m\) since \((a, a_1)_{K(\tilde{a}, \alpha_1)}\). Hence the specialization \(\alpha_{K(\tilde{a}, \alpha_1)}\) is generic, and

\[
K(\tilde{a}, \alpha_1) \cong K(\tilde{a}, \tilde{a}_1).
\]

Using this isomorphism and (4), one obtains

\[
\text{ord } \mathcal{M} = \text{t.d. } K(\tilde{a})/K = \text{t.d. } K(\tilde{a}_1)/K
\leq \text{t.d. } K(\tilde{a}, \tilde{a}_1)/K - \text{t.d. } K(\tilde{a})/K + \text{t.d. } K(a_1)/K
\leq 0 + n - \text{t.d. } K(a, a_1)/K.
\]

But, by the same argument as in Theorem 3 of [2], \(\text{t.d. } K(a, a_1)/K(\alpha) \geq n - J(A)\). Therefore, \(\text{ord } \mathcal{M} \leq J(A)\).

By an example analogous to that of [2], it can be shown that for any matrix \(A\) of nonnegative integers, there exists a system of \(n\) difference polynomials in \(n\) indeterminates with matrix of orders equal to \(A\) such that some zero dimensional component of the variety of the system has effective order equal to \(J(A)\). Thus when the Jacobi number is a bound, it is a bound achieved by some system.

**References**


**Department of Mathematics, University of Alaska, College, Alaska 99701**