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THE ABSCISSA OF ABSOLUTE SUMMABILITY OF LAPLACE INTEGRALS

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ABSTRACT. With $A(u)$ of bounded variation over every finite interval of the nonnegative real axis, we write $C(w)=\int_0^w e^{-us}\,dA(u)$ and (formally)

$$R(k', w) = (\Gamma(k' + 1))^{-1}\int_w^\infty (u - w)^{k'}\, dA(u) \qquad (k' \geqq 0).$$

It is shown that if k is positive and fractional and if $e^{-ws'}R(k, w)$ is summable $|C, 0|$ for some s' whose real part is negative, then $C(w)$ is summable $|C, k+\varepsilon|$ for each $\varepsilon>0$, where s is such that its real part is greater than that of s'; if k is nonnegative and integral the result holds with $\varepsilon=0$. Together with a 'converse' result, this may be used to show that if the abscissa of $|C, k|$ summability of $\int_0^\infty e^{-us}\,dA(u)$ is negative then it equals

$$\limsup_{w\to\infty} w^{-1} \log \int_w^\infty |dR(k, u)|$$

for all $k\geqq 0$ except one fractional value.

1. If

$$\Gamma(k + 1)x^{-k}F_k(a; x) \equiv x^{-k}\int_a^x (x - u)^k f(u)\, dA(u) = L + o(1)$$

as $x\to\infty$ (or is of bounded variation over $[a, \infty)$, with limit L), where $A(u)$ is of bounded variation over every finite interval of the nonnegative real axis and the integral is taken in the Riemann-Stieltjes sense, we say that $F(a; x)$ (i.e., $F_0(a; x)$) *is summable* (C, k) (*or* $|C, k|$) *to* L. If now, $(\Gamma(k' + 1))^{-1}\int_w^x (u - w)^{k'}\, dA(u)$ is summable (C, k) (or $|C, k|$) to L, we shall say that

$$R(k', w) \equiv (\Gamma(k' + 1))^{-1}\int_w^\infty (u - w)^{k'}\, dA(u)$$

exists in the (C, k) (*or* $|C, k|$) *sense and has value* L. The notation $h(x)= L+o(1)\ |C, k|$ will mean

$$\int_1^x dh(u) = L - h(1) + o(1)\ |C, k|.$$

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We shall write V for the class of functions of bounded variation over $[1, \infty)$; $[k]$ for the largest integer less than or equal to k, $\langle k\rangle$ for $k-[k]$; c, c_r, c'_r for constants; and

$$C_k(x) = (\Gamma(k+1))^{-1}\int_0^x (x-u)^k e^{-us}\, dA(u) \quad (k \geqq 0),\ C(x) = C_0(x),$$

where s is complex and $\mathrm{Re}(s)=\sigma$. We have now,

THEOREM A [6]. *If k is positive and fractional, and if $C(w)$ is summable $|C, k|$ for some s such that $\sigma<0$, then $R(k+\delta, w)$ exists in the $|C, k|$ sense and*

$$e^{-ws}w^{-k}R(k+\delta, w) = o(1)\ |C, 0| \quad \textit{for each } \delta > 0.$$

THEOREM B. *If k is positive and fractional, and if $R(k, w)$ exists in the (C, p) sense for some $p\geqq k$ and satisfies that $e^{-ws'}R(k, w)$ is summable $|C, 0|$ for some s' such that $\mathrm{Re}(s')=\sigma'<0$, then $C(w)$ is summable $|C, k+\varepsilon|$ whenever $\sigma>\sigma'$, for every $\varepsilon>0$.*

Theorem B is proved below. For k integral, by [3, Theorem 3] and an easy version of the proof below, we see that Theorems A and B are true with $\delta=0$, $\varepsilon=0$ respectively. The (C) analogues in all cases are substantially given in [4]. For a general discussion of related results see [8, p. 10].

2. **Proof of Theorem B.** This will follow from

LEMMA 1. *Under the hypotheses of Theorem B, if $0\leqq\varepsilon<1-\langle k\rangle$, then for $\sigma>\sigma'$,*

$$w^{-k-\varepsilon}C_{k+\varepsilon}(w) + (-1)^{[k]}w^{-k-\varepsilon}e^{-ws}U^{(\langle k\rangle+\varepsilon)}(w) \quad \textit{is in } V,$$

where

$$U^{(r)}(w) = (\Gamma(r))^{-1}\int_{w-1}^{w} (w-u)^{r-1}R([k], u)\, du \qquad (r > 0).$$

LEMMA 2. *Under the hypotheses of Theorem B,*

$$e^{-ws'}U^{(\langle k\rangle+\varepsilon)}(w) \quad \textit{is in } V \quad \textit{for each } \varepsilon > 0.$$

PROOF OF LEMMA 1. *Case $\varepsilon=0$.* We may take A continuous on the right. Then by [5, Theorem 2], and [4, Lemma 2, Corollary],

$$R([k], u) = (\Gamma(1-\langle k\rangle))^{-1}\left(\int_u^{u+1} + \int_{u+1}^{\infty}\right)(t-u)^{-\langle k\rangle}\, dR(k, t);$$

we see that the integral of $R([k], u)$ from w to ∞ is convergent, and hence by [5, p. 236 line -2],

(1) $\qquad R([k]+1, u)$ exists in the (C, p) sense.

We now write

$$(2)\qquad \text{(a) } R(k,t)=e^{ts'}f(t); \qquad \text{(b) } R([k]+1,t)=e^{ts'}g(t).$$

Then by hypothesis, $f(t)$ is in V. By [4, Lemma 2],

$$(3)\qquad R([k]+1,w)=(\Gamma(1-\langle k\rangle))^{-1}\int_w^\infty (t-w)^{-\langle k\rangle}R(k,t)\,dt,$$

the integral being convergent by our hypothesis. Inserting (2)(a), then putting $t=w+x$ and applying [6, Lemma 2], we see that $g(w)$ is in V. By [4, (64)–(70)] we have

$$(4)\qquad C_k(w)=\sum_{r=0}^{[k]}c_r w^{k-r}+\sum_{r=0}^{[k]+1}c_r' I_r=A+B,$$

say, where

$$I_r=\int_0^w R([k],u)(w-u)^{k-r}e^{-us}\,du.$$

Writing $R([k],u)$ as the derivative of $-e^{us'}g(u)$ (by (1) and [4, Lemma 2, Corollary]) we now obtain

$$(5)\qquad w^{-k}I_r=-w^{-r}(s'N_1(r,w)+N_2(r,w)) \qquad (r=0,1,\cdots,[k])$$

where

$$N_1(r,w)=\int_0^w (1-u/w)^{k-r}e^{-u(s-s')}g(u)\,du$$

and $N_2(r,w)$ is the same integral with g replaced by g'. Also

$$I_{[k]+1}=\left(\int_0^{w-1}+\int_{w-1}^{w}\right)R([k],u)(w-u)^{\langle k\rangle-1}e^{-us}\,du=P+Q,$$

say. But

$$(6)\qquad P=-w^{\langle k\rangle-1}(s'N_1([k]+1,w-1)+N_2([k]+1,w-1))$$

and Q may be expressed as

$$(7)\qquad \begin{aligned} &e^{-ws}\int_{w-1}^{w}R([k],u)(w-u)^{\langle k\rangle-1}\,du \\ &\qquad+\int_{w-1}^{w}R([k],u)(w-u)^{\langle k\rangle-1}(e^{-us}-e^{-ws})\,du=Q_1+Q_2, \end{aligned}$$

say. An integration by parts of Q_2, followed by the insertion of (2)(b) and then the substitution $u=w-x$, gives

$$Q_2 = (1 - e^{-s})e^{-(w-1)(s-s')}g(w-1) - s\int_0^1 x^{\langle k\rangle -1}e^{-(w-x)(s-s')}g(w-x)\,dx$$
$$- (\langle k\rangle - 1)\int_0^1 x^{\langle k\rangle -2}(1 - e^{-sx})e^{-(w-x)(s-s')}g(w-x)\,dx.$$

Applying [**4**, Lemma 2] to the last two integrals, and also to those in (6) and (5), shows that Q_2 and P are in V and that $w^{-k}I_r$ is in V ($r=0, 1, \cdots, [k]$). Since Q_1 is just $\Gamma(\langle k\rangle)e^{-ws}U^{(\langle k\rangle)}(w)$ and (in (4)) $c'_{[k]+1}=(-1)^{[k]+1}(\Gamma(\langle k\rangle))^{-1}$, this completes the proof.

Case $\varepsilon>0$. By (1) and [**1**, p. 300], $R(k+\varepsilon, w)$ certainly exists in the (C,p) sense, and thus by [**4**, Lemma 2], (3) holds with $R([k]+1, w)$ and $-\langle k\rangle$ replaced by $R(k+\varepsilon, w)$ and $\varepsilon-1$ respectively. Hence $e^{-ws'}R(k+\varepsilon, w)$ is in V. The required result now follows from the previous case with k replaced by $k+\varepsilon$.

PROOF OF LEMMA 2. We write $S(u, w)=R([k]+1, w)-R([k]+1, u)$. Then by (3) and (2)(a), $-\Gamma(1-\langle k\rangle)S(u, w)$ equals

$$(8)\qquad \int_u^w (t-u)^{-\langle k\rangle}e^{ts'}f(t)\,dt + \int_w^\infty \{(t-u)^{-\langle k\rangle} - (t-w)^{-\langle k\rangle}\}e^{ts'}f(t)\,dt.$$

Now by an integration by parts, $e^{-ws'}U^{(\langle k\rangle+\varepsilon)}(w)$ has value

$$c_1e^{-ws'}S(w-1, w) + c_2e^{-ws'}\int_{w-1}^w S(u, w)(w-u)^{\langle k\rangle+\varepsilon-2}\,du = G + H,$$

say. By (2)(b), G is in V. We now insert (8) in H, obtaining H_1+H_2, say. In H_1 we put $t=w-y$ and then $u=w-x$; in H_2 we put $t=w+v$, $u=w-x$. Applying [**6**, Lemma 2], we see that H_1 and H_2 are in V. This completes the proof.

3. **The abscissa $\bar\sigma_k$.** We shall write $\bar\sigma_k$ for the infimum of the set of σ's for which $C(w)$ is summable $|C, k|$, and $\bar k$ for the infimum of the set of k's for which $\bar\sigma_k$ is less than infinity. It is known (see [**9**], [**7**], [**2**, Lemma 13]) that $\bar\sigma_k$ is continuous for $k>\bar k$, the value $-\infty$ being allowed. We have now, as a deduction from Theorems A and B (compare [**4**, pp. 470, 475]):

THEOREM A*. *If k is positive and fractional, $k>\bar k$ and $C(w)$ is summable $|C, k|$ for some s such that $\sigma<0$, then $R(k, w)$ exists in the $|C, k|$ sense and $e^{-ws'}R(k, w)=o(1)$ $|C, 0|$ for $\sigma'>\sigma$.*

THEOREM B*. *If k is positive and fractional, $k>\bar k$ and if $R(k, w)$ exists in the (C,p) sense for some $p\geqq k$, and*

(9) $\qquad e^{-ws'}R(k, w)$ *is summable* $|C, 0|$ *for some s' such that* $\sigma' < 0$,

then $C(w)$ is summable $|C, k|$ for every $\sigma>\sigma'$.

We observe that the conclusion of Theorem A* implies

$$\int_w^\infty |d_w R(k, w)| = O(e^{w\sigma'}) \qquad (\sigma' > \sigma) \tag{10}$$

and it is not difficult to see that (10), together with the condition

$$R(k, w) \to 0 \quad \text{as } w \to \infty \tag{11}$$

implies the hypothesis (9) of Theorem B*, with a slightly larger σ'. Since certainly $R(k, w)$ exists in the (C, k) sense (by [**1**, p. 300]) and also (11) holds (by Theorem A*) *if we assume that* $\bar{\sigma}_k$ *is negative*, we obtain the following formula for $\bar{\sigma}_k$ in the case k fractional (see [**4**, p. 463]), a similar argument yielding the case k integral:

THEOREM C. *Let* $\bar{\sigma}_k$ *be negative. Then if* $k=0, 1, \cdots$, *or if* k *is fractional and* $k>\bar{k}$,

$$\bar{\sigma}_k = \limsup_{w\to\infty} w^{-1} \log \int_w^\infty |dR(k, u)|. \tag{12}$$

If k *is fractional and* $k=\bar{k}$, *it is possible for the formula to fail.*

For the second part we choose the function $A(u)$ used in the proof of [**6**, Theorem A$'''$]; then the right side of (12) is infinity.

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