PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 32, Number 1, March 1972

A SMALL BOUNDARY FOR H^∞ ON THE POLYDISC[1]

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ABSTRACT. Let Δ^n be the unit polydisc in C^n and let T^n be its distinguished boundary. It is shown that for $n \geqq 2$ there is a nowhere dense subset of the maximal ideal space of $L^\infty(T^n)$ which defines a closed boundary for $H^\infty(\Delta^n)$.

Let $H^\infty(\Delta^n)$ be the Banach algebra of bounded holomorphic functions on the open unit polydisc $\Delta^n = \{|z_i| < 1,\ 1 \leqq i \leqq n\}$ in C^n. Denote by $T^n = \{|z_i| = 1,\ 1 \leqq i \leqq n\}$ the distinguished boundary of Δ^n, and let σ be the area measure on T^n. By taking radial limits, each $f \in H^\infty(\Delta^n)$ defines almost everywhere on T^n a function $f^* \in L^\infty(T^n, \sigma)$; $H^\infty(\Delta^n)$ can thus be identified with a closed subalgebra of $L^\infty(\sigma)$ (see, for example, W. Rudin [3]).

Let $\mathcal{M}_n$ and X_n be the maximal ideal spaces of $H^\infty(\Delta^n)$ and $L^\infty(\sigma)$ respectively. A *closed boundary* for $H^\infty(\Delta^n)$ is a closed subset Γ of $\mathcal{M}_n$, such that $\|f\| = \sup\{|\gamma(f)| : \gamma \in \Gamma\}$ for all $f \in H^\infty(\Delta^n)$. The above identification gives rise to a continuous map $\tau : X_n \to \mathcal{M}_n$ defined by

$$\tau(\varphi)(f) = \varphi(f^*) \quad \text{for } \varphi \in X_n \text{ and } f \in H^\infty(\Delta^n),$$

whose image $\tau(X_n)$ is a closed boundary for $H^\infty(\Delta^n)$.

It is known that for $n = 1$, the map $\tau : X_1 \to \mathcal{M}_1$ is actually a homeomorphism from X_1 onto the Shilov boundary (i.e. the smallest closed boundary) of $H^\infty(\Delta^1)$ (see K. Hoffman [**2**, p. 174]).

The purpose of this note is to show that the corresponding result is no longer true in higher dimensions. This will be accomplished by constructing a "very small" subset of X_n which maps onto a boundary. The precise statement is as follows.

THEOREM. *Let $n \geqq 2$. There is a closed, nowhere dense subset β of X_n with measure $\hat{\sigma}(\beta) = 0$ such that $\tau(\beta)$ is a closed boundary for $H^\infty(\Delta^n)$.*

Received by the editors April 30, 1971.

AMS 1970 *subject classifications*. Primary 32E25, 46J20.

Key words and phrases. Bounded holomorphic functions, polydisc, closed boundary, Shilov boundary.

[1] Research supported by NSF Grant GP-11475.

Here $\hat{\sigma}$ denotes the unique regular Borel measure on X_n which satisfies

$$\int_{T^n} g\, d\sigma = \int_{X_n} \hat{g}\, d\hat{\sigma}$$

for all $g \in L^\infty(\sigma)$ ($\hat{g}$ denotes the Gelfand transform of g).

REMARKS. An analogous result is valid for the unit ball in $\boldsymbol{C}^n$. It seems to be an open problem to characterize the Shilov boundary of $H^\infty(\Delta^n)$. The above result does not give information on whether τ is one-to-one or $\tau(X_n)$ is the Shilov boundary. However, it shows that at most one of these statements can be true.

PROOF. For simplicity, we will only consider the case $n=2$. The general case can be handled similarly. The construction of β will be done in three steps. First, one constructs a suitable sequence $\{E_k\}_{k=1}^\infty$ of subsets of T^2. Then one sets $U_k = \{\varphi \in X_2 : \varphi(\chi_{E_k}) = 1\}$, and one shows that $\tau(U_k)$ is a boundary. Finally, one verifies that $\beta = \bigcap_{k=1}^\infty U_k$ has all the desired properties.

Step 1. It will be convenient to parametrize the torus T^2 in such a way that the circles $\{e^{i\theta} w : 0 \leqq \theta \leqq 2\pi\}$ for $w \in T^2$ correspond to lines parallel to a coordinate axis. Thus, let $Q = [0, \pi] \times [0, 2\pi] \subset \boldsymbol{R}^2$, and define the continuous map $\rho : Q \to T^2$ by

$$\rho(x, y) = (e^{i(x+y)}, e^{i(-x+y)}).$$

ρ is onto, $\rho(\partial Q)$ has measure 0, and ρ is a diffeomorphism from the interior of Q onto $T^2 - \rho(\partial Q)$. If $w_0 = \rho(x_0, y_0)$ is in T^2, then

$$(*) \qquad \{e^{i\theta} w_0 : 0 \leqq \theta \leqq 2\pi\} = \rho(\{(x_0, y) : 0 \leqq y \leqq 2\pi\}).$$

Let $\{r_j\}_{j=1}^\infty$ be an enumeration of the rational numbers in $(0, \pi)$. Fix a positive integer k, let $I_k^{(j)} = \{x \in (0, \pi) : |x - r_j| < (1/2k)(1/2^{j+1})\}$ for $j = 1, 2, \cdots$, and set $I_k = \bigcup_{j=1}^\infty I_k^{(j)}$. One verifies easily that $E_k = \rho(I_k \times [0, 2\pi])$ is an open, dense subset of T^2 with $\sigma(E_k) < 1/k$, and from (*) it follows that

$$(**) \qquad w \in E_k \quad \text{implies} \quad \{e^{i\theta} w : 0 \leqq \theta \leqq 2\pi\} \subset E_k.$$

Step 2. Let $U_k = \{\varphi \in X_2 : \varphi(\chi_{E_k}) = 1\}$, where χ_{E_k} is the characteristic function of E_k. U_k is a closed open subset of X_2 with $\hat{\sigma}(U_k) = \sigma(E_k) < 1/k$ (see T. W. Gamelin [**1**, Chapter I]). In order to show that the closed set $\tau(U_k) \subset \mathscr{M}_2$ is a boundary, we define, given $f \in H^\infty(\Delta^2)$, a function $G_f : T^2 \to \boldsymbol{R}$ by

$$G_f(w) = \sup_{\lambda \in \Delta} |f(\lambda w)| = \operatorname{ess\,sup}_{|\alpha|=1} |f^*(\alpha w)|$$

for $w \in T^2$. One shows easily that G_f is lower semicontinuous (Rudin [**3**, Theorem 3.5.2]).

Now let $f \in H^\infty(\Delta^2)$, and assume that $\sup_{\tau(U_k)} |\hat{f}| = \operatorname{ess\,sup}_{E_k} |f^*| \leqq 1$. Denote by m_i Lebesgue measure in $\boldsymbol{R}^i$, for $i=1, 2$. The function $g=|f^*|\circ\rho$ is in $L^\infty(Q, m_2)$, and $g \leqq 1$ m_2-a.e. on $I_k \times [0, 2\pi]$. This implies that for m_1-almost all $x \in I_k$ one has $\operatorname{ess\,sup}_{0\leqq y\leqq 2\pi} g(x, y) \leqq 1$, and hence, by (*) and (**)

$$G_f(w) = \operatorname*{ess\,sup}_{|\alpha|=1} |f^*(\alpha w)| \leqq 1 \quad \sigma\text{-a.e. on } E_k.$$

Since G_f is lower semicontinuous, and since E_k is open and dense in T_2, it follows that $G_f(w) \leqq 1$ for all $w \in T^2$, which implies $\|f\| \leqq 1$. Thus, $\tau(U_k)$ is a closed boundary for $H^\infty(\Delta^2)$.

Step 3. The construction of E_k shows that $E_k \subset E_{k+1}$, and hence also $U_k \subset U_{k+1}$ for $k=1, 2, \cdots$. A standard compactness argument then shows that $\beta = \bigcap_{k=1}^\infty U_k$ is nonempty and that $\tau(\beta) = \bigcap_{k=1}^\infty \tau(U_k)$.

Therefore, being an intersection of closed boundaries, $\tau(\beta)$ is a closed boundary as well. Clearly $\hat{\sigma}(\beta)=0$, and since the closed support of $\hat{\sigma}$ is X_2 [**1**, I.9.2], β has no interior. Thus β has all the required properties, and the theorem is proved.

Note. It was remarked by J. P. Rosay that a simple modification of the above argument shows that $\tau: X_2 \to \mathscr{M}_2$ is not one-to-one. One replaces the set $I_k \subset (0, \pi)$ in Step 1 by a measurable set $A \subset (0, \pi)$ satisfying $0 < m_1(A \cap V) < m_1(V)$ for all open sets $V \subset (0, \pi)$. As in Step 2, it follows that $E = \rho(A \times [0, 2\pi])$ and $T^2 - E$ define two disjoint sets in X_2 which map onto a boundary for $H^\infty(\Delta^2)$.

Added in proof. J. P. Rosay observed that the results of this paper imply that $\tau(X_2)$ is strictly larger than the Shilov boundary. In fact, let E be any of the sets constructed in Step 1, and let $F \subset T^2 - E$ be a closed set with positive measure. It follows from [3, 3.5.3] that there is $f \in H^\infty(\Delta^2)$ with $|f^*|=1$ on F and $|f^*|=2$ on E. Thus $|\hat{f}|=2$ on the Shilov boundary, but $|\hat{f}| \not\equiv 2$ on $\tau(X_2)$.

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