SURFACES OF REVOLUTION WITH MONOTONIC INCREASING CURVATURE AND AN APPLICATION TO THE EQUATION $\Delta u = 1 - Ke^{2u}$ ON $S^2$

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Abstract. The geometric result that a compact surface of revolution in $R^3$ cannot have monotone increasing curvature is proved and applied to show that the equation $\Delta u = 1 - Ke^{2u}$, on $S^2$, has no axially symmetric solutions $u$, given axially symmetric data $K$.

1. We shall first show that a compact surface of revolution in $R^3$ cannot have monotone increasing curvature. In §2 we shall use this geometric result to prove a nonexistence result for the equation $\Delta u = 1 - Ke^{2u}$ on $S^2$. Consider a compact surface of revolution in $R^3$ obtained by revolving the profile curve $t \rightarrow (g(t), h(t), 0)$, for $0 \leq t \leq l$, about the first coordinate axis, where necessarily $h(0) = 0 = h(l)$ and $h(t) > 0$ for $0 < t < l$. If the curve is parametrized by arc length, then $h'(0) = 1$, $h'(l) = -1$, and the curvature $K$ of the surface of revolution as a function of $t$ satisfies the equation [4]:

$$h''(t) + K(t)h(t) = 0.$$ 

We shall now derive an integrability condition for solutions of this equation with the above boundary conditions. This condition is not satisfied if $K$ is monotone. (Although we will not need this fact, observe that the monotone condition implies that $K$ is nonnegative, for at the two poles of any compact surface of revolution the curvature is necessarily nonnegative.)

Now, using the differential equation for $h$ we find

$$(h'^2 + Kh^2)' = 2h'(h'' + Kh) + K'h^2 = K'h^2.$$ 

Therefore, because of the boundary conditions on $h$ and $h'$, 

$$\int_0^l K'h^2 \, dt = 0.$$ 

This integrability condition is evidently not satisfied if $K' \geq 0$, unless $K = \text{const}$.

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139
2. Application to the equation $\Delta u = 1 - Ke^{2u}$ on $S^2$. The above elementary fact concerning surfaces of revolution can be used to prove the nonexistence of certain types of solutions of the equation $\Delta u = 1 - Ke^{2u}$ on the ordinary 2-sphere $S^2$. Here $\Delta$ is the Laplace-Beltrami operator relative to the standard metric on $S^2$. Interest in this equation lies in the fact that this is precisely the equation which describes the change in curvature of the 2-sphere under a conformal change $e^{2ug}$ of the standard metric $g$. More generally, if $M$ is any two dimensional Riemannian manifold with Riemannian metric $g$ and Gaussian curvature $K_g$, then the curvature $K$ of the metric $e^{2ug}$ on $M$ satisfies $\Delta K = K_g - Ke^{2u}$.

We consider a fixed axis of $S^2$ and all those $C^\infty$ functions on $S^2$ which are invariant under rotation about this fixed axis. For simplicity, we shall use "rotationally symmetric function" to refer to one of these $C^\infty$ functions. We shall show that if $K$ is a rotationally symmetric function which is strictly positive and monotone increasing, then the equation $\Delta u = 1 - Ke^{2u}$ has no rotationally symmetric solution. For suppose that $u$ is a rotationally symmetric solution. Then if $g$ is the standard metric on $S^2$, $e^{2ug}$ is a Riemannian metric on $S^2$ with curvature $K$. Now the metric $e^{2ug}$ is invariant under rotations about our fixed axis. Therefore, these rotations determine a one-parameter group $\beta_t$ of isometries of $S^2$ ($t \in S^1$). Let $\alpha$ be an isometric imbedding of $S^2$ with the metric $e^{2ug}$ in $R^3$. That such an imbedding exists depends on the positivity of $K$ and follows from the existence part of the famous Weyl problem (see [3]). By composing with $\beta_t$, we obtain a one-parameter family $\alpha_t = \alpha \circ \beta_t$ of isometric imbeddings of $(S^2, e^{2ug})$ into $R^3$. Let $p$ be one of the poles of our fixed axis of $S^2$. Then the point $\alpha_t(p)$ and the tangent plane $d\alpha_t(S^2_p)$ are independent of $t$. It follows from the uniqueness theorem of Cohn-Vossen [1] that the various imbeddings $\alpha_t$ differ only by rotations about an axis through the point $\alpha_t(p)$ and orthogonal to the tangent plane $d\alpha_t(S^2_p)$. Therefore $\alpha(S^2)$ is a surface of revolution in $R^3$. But this surface of revolution is compact and has monotone increasing curvature which, as we have seen, is impossible. Thus we have the nonexistence of rotationally symmetric solutions $u$.

Observe that this says nothing concerning the existence of nonrotationally symmetric solutions $u$. However, in an attempt to find a purely analytic, nongeometric proof of the above fact, we have been led to a new integrability condition for the above equation which shows, in particular, that for positive, monotone increasing, rotationally symmetric functions $K$, there are no solutions whatever of $\Delta u = 1 - Ke^{2u}$ on $S^2$. For this see [2].

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