PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 32, Number 1, March 1972

SURFACES OF REVOLUTION WITH MONOTONIC INCREASING CURVATURE AND AN APPLICATION TO THE EQUATION $\Delta u=1-Ke^{2u}$ ON S^2

JERRY L. KAZDAN AND FRANK W. WARNER[1]

ABSTRACT. The geometric result that a compact surface of revolution in R^3 cannot have monotonic increasing curvature is proved and applied to show that the equation $\Delta u=1-Ke^{2u}$, on S^2, has no axially symmetric solutions u, given axially symmetric data K.

1. We shall first show that *a compact surface of revolution in R^3 cannot have monotonic increasing curvature*. In §2 we shall use this geometric result to prove a nonexistence result for the equation $\Delta u=1-Ke^{2u}$ on S^2. Consider a compact surface of revolution in R^3 obtained by revolving the profile curve $t\mapsto(g(t), h(t), 0)$, for $0\leqq t\leqq l$, about the first coordinate axis, where necessarily $h(0)=0=h(l)$ and $h(t)>0$ for $0<t<l$. If the curve is parametrized by arc length, then $h'(0)=1$, $h'(l)=-1$, and the curvature K of the surface of revolution as a function of t satisfies the equation **[4]**:

$$h''(t) + K(t)h(t) = 0.$$

We shall now derive an integrability condition for solutions of this equation with the above boundary conditions. This condition is not satisfied if K is monotone. (Although we will not need this fact, observe that the monotone condition implies that K is nonnegative, for at the two poles of any compact surface of revolution the curvature is necessarily nonnegative.)

Now, using the differential equation for h we find

$$(h'^2 + Kh^2)' = 2h'(h'' + Kh) + K'h^2 = K'h^2.$$

Therefore, because of the boundary conditions on h and h',

$$\int_0^l K'h^2\,dt = 0.$$

This integrability condition is evidently not satisfied if $K'\geqq 0$, unless $K\equiv$const.

Received by the editors April 1, 1971.
AMS 1970 *subject classifications*. Primary 35J60, 53C45; Secondary 53A30, 53G20.
[1] This research was supported, in part, by NSF GP 19693.

2. **Application to the equation $\Delta u = 1 - Ke^{2u}$ on S^2.** The above elementary fact concerning surfaces of revolution can be used to prove the nonexistence of certain types of solutions of the equation $\Delta u = 1 - Ke^{2u}$ on the ordinary 2-sphere S^2. Here Δ is the Laplace-Beltrami operator relative to the standard metric on S^2. Interest in this equation lies in the fact that this is precisely the equation which describes the change in curvature of the 2-sphere under a conformal change $e^{2u}g$ of the standard metric g. More generally, if M is any two dimensional Riemannian manifold with Riemannian metric g and Gaussian curvature K_g, then the curvature K of the metric $e^{2u}g$ on M satisfies $\Delta_g u = K_g - Ke^{2u}$.

We consider a fixed axis of S^2 and all those C^∞ functions on S^2 which are invariant under rotation about this fixed axis. For simplicity, we shall use "rotationally symmetric function" to refer to one of these C^∞ functions. We shall show that *if K is a rotationally symmetric function which is strictly positive and monotone increasing, then the equation $\Delta u = 1 - Ke^{2u}$ has no rotationally symmetric solution*. For suppose that u is a rotationally symmetric solution. Then if g is the standard metric on S^2, $e^{2u}g$ is a Riemannian metric on S^2 with curvature K. Now the metric $e^{2u}g$ is invariant under rotations about our fixed axis. Therefore, these rotations determine a one-parameter group β_t of isometries of S^2 ($t \in S^1$). Let α be an isometric imbedding of S^2 with the metric $e^{2u}g$ in R^3. That such an imbedding exists depends on the positivity of K and follows from the existence part of the famous Weyl problem (see [3]). By composing with β_t we obtain a one-parameter family $\alpha_t = \alpha \circ \beta_t$ of isometric imbeddings of $(S^2, e^{2u}g)$ into R^3. Let p be one of the poles of our fixed axis of S^2. Then the point $\alpha_t(p)$ and the tangent plane $d\alpha_t(S_p^2)$ are independent of t. It follows from the uniqueness theorem of Cohn-Vossen [1] that the various imbeddings α_t differ only by rotations about an axis through the point $\alpha_t(p)$ and orthogonal to the tangent plane $d\alpha_t(S_p^2)$. Therefore $\alpha(S^2)$ is a surface of revolution in R^3. But this surface of revolution is compact and has monotone increasing curvature which, as we have seen, is impossible. Thus we have the nonexistence of rotationally symmetric solutions u.

Observe that this says nothing concerning the existence of nonrotationally symmetric solutions u. However, in an attempt to find a purely analytic, nongeometric proof of the above fact, we have been led to a new integrability condition for the above equation which shows, in particular, that for positive, monotone increasing, rotationally symmetric functions K, there are no solutions whatever of $\Delta u = 1 - Ke^{2u}$ on S^2. For this see [2].

References

1. S. S. Chern, *Studies in global geometry and analysis*, Studies in Mathematics, vol. 4, Prentice-Hall, Englewood Cliffs, N.J., 1967, p. 46. MR **35** #1429.

2. J. Kazdan and F. Warner, *Integrability conditions for $\Delta u = k - Ke^{\alpha u}$ with applications to Riemannian geometry*, Bull. Amer. Math. Soc. **77** (1971), 819–823.

3. L. Nirenberg, *The Weyl and Minkowski problems in differential geometry in the large*, Comm. Pure Appl. Math. **6** (1953), 337–394. MR **15**, 347.

4. B. O'Neill, *Elementary differential geometry*, Academic Press, New York, 1966. MR **34** #3444.

Department of Mathematics, University of Pennsylvania, Philadelphia, Pennsylvania 19104