CRITICALLY \( n \)-CONNECTED GRAPHS

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Abstract. The following result is proved. Every \( n \)-connected graph contains either a vertex whose removal results in a graph which is also \( n \)-connected or a vertex of degree less than \((3n-1)/2\).

Introduction. A graph \( G \) is said to be \( n \)-connected if the removal of fewer than \( n \) vertices from \( G \) neither disconnects it nor reduces it to the trivial graph consisting of a single vertex. The maximum value of \( n \) for which a graph \( G \) is \( n \)-connected is called its connectivity and is denoted by \( \kappa(G) \). The minimum degree of \( G \) is designated by \( \delta(G) \); the inequality \( \kappa(G) \leq \delta(G) \) is well known.

A graph \( G \) is said to be critically \( n \)-connected if \( \kappa(G) = n \) and \( \kappa(G-v) = n-1 \) for each vertex \( v \) of \( G \). Analogously, a graph \( G \) is minimally \( n \)-connected if \( \kappa(G) = n \) and for each edge \( e \) of \( G \), \( \kappa(G-e) = n-1 \). The object of this article is to present a necessary condition for a graph to be critically \( n \)-connected and to discuss related topics.

Since 1-connected graphs are the nontrivial connected graphs and since every nontrivial connected graph \( G \) has at least two vertices \( u \) and \( v \) such that...
that each of $G-u$ and $G-v$ is connected, it follows that the only critically 1-connected graph is the complete graph of order two. It is also easily observed that a graph is minimally 1-connected if and only if it is a non-trivial tree; thus if $G$ is a graph which is either critically 1-connected or minimally 1-connected, then $\delta(G)=1$. Dirac [2] and Plummer [7] have shown that if $G$ is minimally 2-connected then $\delta(G)=2$. Recently, Halin [4] extended this result so that if $G$ is a minimally $n$-connected graph, $n \geq 1$, then $\delta(G)=n$. It was shown in [5] that every critically 2-connected graph has minimum degree 2. The graph in Fig. 1 shows that no theorem on critically $n$-connected graphs analogous to Halin's theorem on minimally $n$-connected graphs is possible. The graph $G$ of Fig. 1 is critically 4-connected but $\delta(G)=5$.

We shall prove that every critically $n$-connected graph, $n \geq 2$, has a vertex of degree less than $(3n-1)/2$ and that the number $(3n-1)/2$ cannot be improved.

**Preliminaries.** Before proceeding further, it is convenient to give a few definitions and establish some notation. All terms not defined here may be found in Harary [3].

If $U$ is a nonempty subset of the vertex set $V(G)$ of $G$, then the *subgraph $H$ induced by $U$*, written $H=\langle U \rangle$, is the subgraph whose vertex set is $U$ and where two vertices are adjacent if and only if these vertices are adjacent in $G$. A set $S$ of vertices of $G$ is called a *cut set of $G$* if the (induced) subgraph $G-S=\langle V(G)-S \rangle$ is disconnected; $S$ is an *$n$-cut set* if $|S|=n$. Two paths of $G$ are said to be *disjoint* if they have no vertices in common except possibly end vertices.

Two special classes of graphs which we shall encounter are the complete graphs and the complete bipartite graphs. The *complete graph $K_p$* has $p$ vertices every two of which are adjacent. The *complete bipartite graph $K(m, n)$* has its vertex set $V$ partitioned into two subsets $V_1$ and $V_2$, where $|V_1|=m$ and $|V_2|=n$, such that two vertices $u$ and $v$ are adjacent if and only if $u \in V_i$ and $v \in V_j$, $i \neq j$.

The concepts of "critically $n$-connected" and "minimally $n$-connected" are independent in the sense that neither property implies the other. For

![Figure 2](image-url)
example, the graph \( G_1 \) of Fig. 2 is critically 2-connected but not minimally 2-connected while \( G_2 = K(2, 3) \) is minimally 2-connected and not critically 2-connected. In general, the graph \( K(n, n+1) \) is minimally \( n \)-connected but not critically \( n \)-connected. For \( n \geq 3 \), the graph obtained by adding an extra edge to \( K(n, n) \) is critically \( n \)-connected but not minimally \( n \)-connected.

We note that it is rarely easy to ascertain whether a given graph is critically \( n \)-connected for some \( n \). Despite this fact, such graphs are quite numerous; indeed if \( G \) is \( n \)-connected and \( G' \) is a subgraph of \( G \) containing the minimum number of vertices such that \( G' \) is \( n \)-connected, then \( G \) is critically \( n \)-connected.

**A necessary condition for critically \( n \)-connected graphs.** We now present the main result of this article.

**Theorem.** If \( G \) is a critically \( n \)-connected graph, \( n \geq 2 \), then \( \delta(G) < (3n-1)/2 \) and the number \( (3n-1)/2 \) cannot be improved.

**Proof.** Suppose the theorem to be false so that there exists a graph \( G \) of order \( p \) having \( \kappa(G) = n \) and \( \delta(G) \geq (3n-1)/2 \) such that for every \( v \in V(G) \), \( \kappa(G-v) = n-1 \). We note that since \( \delta(G) \geq (3n-1)/2 \), \( G \) is not complete. This implies that every vertex of \( G \) belongs to some \( n \)-cut set of \( G \).

Among all \( n \)-cut sets \( S' \) of \( G \), let \( S \) be one such that \( G-S \) contains a component \( G_1 \) of smallest order; denote the order of \( G_1 \) by \( m \). Furthermore, let \( G_2 = G-S-V(G_1) \).

Let \( w \in V(G_1) \) and \( u \in V(G_2) \). By a result of Whitney \([8]\) there exist \( n \) disjoint \( u-v \) paths in \( G \); necessarily, each such path contains precisely one vertex of \( S \). Hence there exist \( n \) disjoint paths joining \( u \) and \( S \) (and also \( v \) and \( S \)).

Let \( w \in V(G_1) \), and let \( S^* \) be an \( n \)-cut set of \( G \) containing \( w \). Define \( G^* = G-S^* \) and, furthermore, let \( V_1 = V(G_1) \cap S^* \), \( V_2 = V(G_2) \cap S^* \), and \( V_3 = S \cap S^* \), where \( |V_i| = n_i \), \( i = 1, 2, 3 \). We note that \( n_1 + n_2 + n_3 = n \) and \( n_1 \geq 1 \).

We now show that \( n_2 \geq n_1 \). If \( S^* \supseteq V(G_2) \), then this is obvious. Assume therefore that \( V(G_2) - V_2 \neq \emptyset \). We have already noted that for each \( u \in V(G_2) \), there exists in \( G \) a set of \( n \) disjoint paths joining \( u \) and \( S \). If \( u \in V(G_2) - V_2 \), then at least \( n-n_2-n_3 = n_1 \) of these paths contain no vertices of \( V_3 \cup V_3 \). In this case, denote the set of end vertices in \( S \) of these \( n_1 \) (or more) paths by \( R(u) \). Thus for each \( u \in V(G_2) - V_2 \), there exists a set \( R(u) \supseteq S - V_3 \) such that there are disjoint paths containing no elements of \( V_3 \cup V_3 \) which join \( u \) and \( R(u) \) where \( |R(u)| \geq n_1 \). If there exist vertices \( u_1, u_2 \in V(G_2) - V_2 \) such that \( R(u_1) \cap R(u_2) = \emptyset \), then \( |S - V_3| \geq 2n_1 \) so that \( n-n_3 \geq 2n_1 \) and \( n_2 \geq n_1 \). Otherwise, let \( R = \bigcup R(u) \), the union taken over all
ueV(G2)−V2, and let G′=(R∪(V(G2)−V2)). It is now easy to verify that every two vertices of G′ are connected so that G′ itself is connected. Hence G′ is a subgraph of a component of G∗. Since the order of G′ is at least \( n_1+(p−m−n)−n_2 \), there must be a component of G∗ of order at most \( m+n_2−n_1 \). Therefore, \( m\leq m+n_2−n_1 \) so that \( n_2\geq n_1 \). Thus in any case, \( n_2\geq n_1 \).

The inequality \( n_2\geq n_1 \) implies that \( n_1\leq n/2 \). We next verify that \( V(G_1)−V_1\neq \emptyset \) or, equivalently, that \( n_1<m \). Assume that \( n_1=m \) so that \( V(G_1)=V_1 \). Hence for each \( v\in V(G_1) \),

\[
\deg v\leq (n_1−1) + n \leq (3n − 2)/2,
\]

which contradicts the fact that \( \delta(G)\geq (3n−1)/2 \). We conclude therefore that \( n_1<m \) and \( V(G_1)=V_1\neq \emptyset \).

Let \( F=((V(G_2)−V_2)\cup(S−V_2)) \). We show that F is disconnected. Suppose, to the contrary, that F is a connected subgraph of G∗. Since G∗ is not connected, \( V(G_2)=V_2\neq \emptyset \). Because each \( v\in V(G_2)−V_2 \) is joined to \( S−V_3 \) by at least \( n_1 \) paths in G∗, it follows that G∗ is connected which is impossible. Thus F is disconnected.

Denote the components of F by \( F_t, t=1, 2, \ldots, k \), where \( k\geq 2 \). Furthermore, for each \( t=1, 2, \ldots, k \), denote by \( W_t \) the set of vertices of \( F_t \) in S, where \( |W_t|=s_t \). We note that each \( W_t\neq \emptyset \); for otherwise there would exist a component of F of order less than \( m \) contained in \( (V(G_2)−V_2) \) which would also be a component of G∗.

We claim that precisely one of the subgraphs \( F_t \) contains elements of \( V(G_2)−V_2 \). Assume this is not the case so that there are two subgraphs \( F_t \) and \( F_j \), \( i\neq j \), containing elements of \( V(G_2)−V_1 \). Let \( W_t=\bigcup W_t, t\neq i \), where \( |W_t|=s_t \). Each of the sets \( V_1\cup V_3\cup W_t \) and \( V_1\cup V_3\cup W_t \) is a cut set of G, for in each case the removal of the set from G produces a graph having a component contained in \( (V(G_2)−V_1) \). This implies that \( n_1+n_2+s_t\geq n \) and \( n_1+n_3+s_t\geq n \) so that \( s_t\geq n_2 \) and \( s_t\geq n_2 \). However, the equality \( n_1+n_2+n_3=s_1+s_t+n_3=n_2 \) together with the inequality \( n_2\geq n_1 \) yield \( s_1=s_1=n_1=n_2 \). Therefore, \( V_1\cup V_3\cup W_t \) is an \( n \)-cut set of G, but the graph \( G−(V_1\cup V_3\cup W_t) \) has a component of order less than \( m \). This produces a contradiction; hence exactly one of the subgraphs \( F_t \) contains elements of \( V(G_2)−V_1 \). Let \( F_1 \) be the subgraph with this property.

Now \( V_1\cup V_3\cup W_1 \) is a cut set of G so that \( n_1+n_3+s_1\geq n \) or \( s_1\geq n_2 \). Let \( G_1^* \) be a component of G∗ which contains vertices of \( W_1 \). If \( V(G_1^*)\subseteq W_1 \), then \( s_1\geq m \), but this implies that

\[
n = s_1 + s_1 + n_3 \geq n_2 + m + n_3 > n_2 + n_1 + n_3 = n,
\]

which is impossible. Therefore, \( G_1^* \) contains vertices of \( V(G_2)−V_2 \), which incidentally shows that \( V(G_2)=V_2 \neq \emptyset \).
We show next that \( V_2 \cup V_3 \cup W_1 \) is a cut set of \( G \). Suppose this is not so. Then \( G' = G - (V_2 \cup V_3 \cup W_1) \) is connected. Since \( F_1 \) is connected, the graph \( G' = G - V_1 \) is also connected. However, \( G* \equiv (V(G')) \cup W_1 \) is disconnected; therefore, \( G* \) has a component which is a subgraph of \( W_1 \), but we have seen that every component of \( G* \) which contains elements of \( W_1 \) also contains elements of \( V(G_2) - V_2 \). Hence \( G - (V_2 \cup V_3 \cup W_1) \) is disconnected so that \( V_2 \cup V_3 \cup W_1 \) is a cut set of \( G \). This produces the inequality

\[ n_2 + n_3 + n_1 \leq n \text{ or } s_1 = s_2. \]

We now know that \( s_1 + s'_1 = n_1 + n_2, \ s_1 \geq n_2, \text{ and } s'_1 \geq n_1 \). From this we conclude that \( s_1 = n_2 \) and \( s'_1 = n_1 \). Returning to the cut set \( V_1 \cup V_3 \cup W_1 \), we note that this is an \( n \)-cut set. However, \( G - (F_1 \cup F_3 \cup F_1) \) contains a component of order less than \( m \). This produces a contradiction, and the desired result follows.

Using the construction in [6], we show that the number \((3n-1)/2\) cannot be improved, i.e., for each positive integer \( n \) and positive integer \( m < (3n-1)/2 \), there is a critically \( n \)-connected graph \( G \) with \( \delta(G) = m \). Before giving the construction, we define the join of two graphs. The join of two graphs \( G_1 \) and \( G_2 \), denoted by \( G_1 \vee G_2 \), is the union \( G_1 \cup G_2 \) together with all edges of the type \( i \backepsilon j \) where \( i \in G_2, \ i = 1,2 \).

For \( n \geq 2m + 2 \), define the collection \( \{H_{n,m}\} \) of graphs as follows:

\[ H_{n,m} = 2K_{m+1}, \quad \text{for } n = 2m + 2, \]

\[ = K_{n-2m-2} + 2K_{m+1}, \quad \text{for } n > 2m + 2. \]

It is easily seen that \( H_{n,m} \) has order \( n \) and \( \delta(H_{n,m}) = n - m - 2 \). Using a result in [1], the equality \( \kappa(H_{n,m}) = n - 2m - 2 \) follows.

For \( n < m < (3n-1)/2 \), define

\[ G_{n,m} = H_{n,m} - n + 2K_{m-n+1}. \]

The graph \( G \) given in Fig. 1 is \( G_{4,5} \). From the information obtained about \( H_{n,m} \), it follows that \( \delta(G_{n,m}) = m \), and with the aid of the above-mentioned result in [1], \( \kappa(G_{n,m}) = n \). Let \( v \) be a vertex of \( G_{n,m} \). If \( v \) belongs to \( H_{n,m} - n \), then the removal of \( v \) and the remaining \( n-1 \) vertices of \( H_{n,m} - n \) results in a disconnected graph; thus, \( \kappa(G_{n,m} - v) = n - 1 \). If \( v \) belongs to \( 2K_{m-n+1} \), then the removal of the vertices of \( 2K_{m-n+1} \) together with a \((3n-2m-2)\)-cut set of \( H_{n,m} - n \) gives a disconnected graph. Hence, here too we have \( \kappa(G_{n,m} - v) = n - 1 \). The graph \( G_{n,m} \) is therefore critically \( n \)-connected.

REFERENCES


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