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COHOMOLOGICAL DIMENSION AND GLOBAL DIMENSION OF ALGEBRAS

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ABSTRACT. Let R be a regular local ring and A an algebra over R which is an R-progenerator. Defining the cohomological dimension of A as $R\text{-dim } A = \text{l}\cdot\text{hd}_{A^e}(A)$, one obtains the Hochschild cohomological dimension of A as an R-algebra. We show the following under the additional hypothesis that R-dim A is finite: (1) $R\text{-dim } A = n$ iff A/N is R-separable and $\text{l}\cdot\text{hd}_A(A/N) = n + \text{gl dim } R$; (2) $\text{gl dim } A = R\text{-dim } A + \text{gl dim } R$; (3) A is R-separable iff $\text{gl dim } A^e = \text{gl dim } R$.

The purpose of this note is to extend a result of Eilenberg on the Hochschild cohomological dimension of associative algebras. The results obtained will relate the global dimension of an algebra A which is a progenerator as an R-module, the global dimension of the ground ring R, and the cohomological dimension of the algebra.

Throughout we will assume that all rings have one. We shall say that an R-algebra A is an R-progenerator in case A is finitely generated, projective, and faithful as a module over the commutative ring R. N will denote the Jacobson radical of the algebra A.

We will always mean by "hd" and "gl dim" the left homological dimension and the left global dimension. Recall that if a ring is commutative or noetherian, the left and right global dimensions coincide. We define $R\text{-dim } A = \text{hd}_{A^e}(A)$, where $A^e = A \otimes_R A^*$ and A^* is the algebra anti-isomorphic to A. Since $A^{e*} = A^e$, its left and right global dimensions also coincide. Hence for the main theorems, global dimension is well defined.

We require two well-known results, the first of Eilenberg, Rosenberg, and Zelinsky [**4**, Proposition 2] and the second due to Kaplansky [**5**, p. 172]:

RESULT 1. If A and B are R-algebras and A is R-flat, then (a) $\text{gl dim } B \otimes A \leqq R\text{-dim } A + \text{gl dim } B$. If further, A is also R-projective and contains R as an R-direct summand, then (b) $\text{gl dim } B \leqq \text{gl dim } B \otimes A$.

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Whenever A is an R-progenerator, R is an R-direct summand of A and hence Result 1 applies in its entirety.

RESULT 2. Let (x) be an ideal of any ring R, x a central nonzero divisor. If M is an $R/(x)$-module of finite homological dimension over $R/(x)$, then $\mathrm{hd}_R(M)=\mathrm{hd}_{R/(x)}(M)+1$.

Combining a result of Auslander [**1**, Corollary 11] with a result of Eilenberg [**3**, Corollary to Theorem III], we have the following well-known

RESULT 3. If A is a finitely generated algebra over a field R, and R-dim A is finite, then R-dim $A=n$ *iff* A/N is R-separable and $n=$ gl dim $A=\mathrm{hd}_A(A/N)$.

Finally, [**7**, Theorem 2.1] shows

RESULT 4. If A is a finitely generated, projective R-algebra and R-dim A is finite, R-dim $A=n$ iff $R/\boldsymbol{m}$-dim $A/\boldsymbol{m}A\leqq n$ for all maximal ideals $\boldsymbol{m}$ of R with equality holding for some $\boldsymbol{m}$. If R is a local ring, A/N is separable and R-dim $A=R/\boldsymbol{a}$-dim $A/\boldsymbol{a}A$ for every proper ideal $\boldsymbol{a}$ of R.

For the remainder of the paper, we shall assume that an R-algebra A is an R-progenerator. If R is a local ring, that A is faithful is a consequence of its being finitely generated and projective.

Recall that a regular local ring R is a commutative noetherian ring with a unique maximal ideal with the additional property that R has finite global dimension. Since every regular local ring is an integral domain, every nonzero element of R is a nonzero divisor of A whenever A is R-projective.

THEOREM A. *Let R be a regular local ring, A an R-progenerator of finite cohomological dimension.* R-dim $A=n$ *iff A/N is R-separable and* $\mathrm{hd}_A(A/N)=n+\mathrm{gl\ dim}\ R$.

PROOF. We proceed by induction on the gl dim R. If gl dim $R=0$, the theorem reduces to Result 3. Suppose it is true for all r less than k. Let gl dim $R=k$. Then there is a principal minimal prime (x) such that gl dim $R/(x)=k-1$ by [**6**, p. 73]. Now R-dim $A/(x)=R/(x)$-dim $A/(x)=$ R-dim $A=n$ *iff* A/N is R-separable and $\mathrm{hd}_{A/(x)}(A/N)=\mathrm{gl\ dim}\ R/(x)+n$ by the inductive hypothesis. Then, by Result 2, this is true *iff* A/N is R-separable and $\mathrm{hd}_A(A/N)=n+\mathrm{gl\ dim}\ R$.

COROLLARY. *If we further assume that either* gl dim R *or* R-dim A *is positive*, R-dim $A=n$ *iff A/N is R-separable and* $\mathrm{hd}_A(N)=\mathrm{gl\ dim}\ R+n-1$.

PROPOSITION 1. gl dim $R\leqq$ gl dim $A\leqq R$-dim $A+$ gl dim R.

PROOF. Apply both parts of Result 1 with (A, B) replaced by (A, R).

PROPOSITION 2. gl dim $R\leqq$ gl dim $A^e\leqq 2$ R-dim $A+$ gl dim R.

PROOF. Apply (a) and (b) with (A, B) replaced by (A^*, A) and (A^e, R) respectively. Note that R-dim $A^*=R$-dim A [**2**, IX, Remark 1, p. 171]. Then apply Proposition 1.

PROPOSITION 3. gl dim $R \leqq$ gl dim $A \leqq$ gl dim A^e.

PROOF. Apply (b) twice, with (A, B) replaced by (A, R) and then by (A^*, A).

THEOREM B. *Let R be a regular local ring, A an R-progenerator of finite cohomological dimension*

$$\text{gl dim } A = R\text{-dim } A + \text{gl dim } R.$$

(One should note that this result is already known for semiprimary algebras over a field and for polynomial algebras.)

PROOF. Recall that R-dim $A=$gl dim $A/\boldsymbol{m}A$ since R is local by Results 3 and 4. We again prove the result by induction on gl dim R. Let (x) be a principal minimal prime such that gl dim $R=$gl dim $R/(x)+1$. Taking suprema over all $A/(x)$-modules in Result 2 together with Proposition 1, we have that gl dim $A/(x)+1 \leqq$ gl dim $A \leqq R$-dim $A+$gl dim R. By the inductive hypothesis, gl dim $A/(x)=R/(x)$-dim $A/(x)+$gl dim $R/(x)=$ R-dim $A+$gl dim $R-1$. Hence it follows that (gl dim $A/(x)+1=$) gl dim $A=R$-dim $A+$gl dim R.

COROLLARY B.1. *Under the same hypotheses, we have that* gl dim $A=\text{hd}_A(A/N)$.

Recall that a ring is said to be *hereditary* if every ideal is projective, but the ring is not semisimple (i.e., the global dimension is one).

COROLLARY B.2. *Under the same hypotheses, we have that A is hereditary iff*

(a) *A is separable and R is a local PID or*

(b) R-dim $A=1$ *and R is a field.*

COROLLARY B.3. *Under the same hypotheses, we have that N is projective as an A-module iff A is R-separable.*

THEOREM C. *Let R be a regular local ring, A an R-progenerator of finite cohomological dimension*

$$\text{gl dim } A^e = 2\, R\text{-dim } A + \text{gl dim } R.$$

PROOF. By Theorem B, gl dim $A^e=R$-dim A^e+gl dim R. Applying Theorem 7.4 [**2**, IX] together with Result 4, one obtains

$$\begin{aligned} R\text{-dim } A^e &= R/\boldsymbol{m}\text{-dim } (A/\boldsymbol{m}A)^e \\ &= R/\boldsymbol{m}\text{-dim } A/\boldsymbol{m}A + R/\boldsymbol{m}\text{-dim } A^*/\boldsymbol{m}A^* = 2\, R\text{-dim } A. \end{aligned}$$

(One should notice that it is easy to obtain the inequality gl dim $R+$ R-dim $A \leqq$ gl dim $A^e \leqq 2$ R-dim $A+$gl dim R from Theorem B together with the left-hand inequalities of Propositions 2 and 3.)

COROLLARY C.1. *Under the same hypotheses*, gl dim $A^e=$gl dim R *iff* A *is* R*-separable.*

This corollary is an interesting extension of a theorem contained in Cartan and Eilenberg [**2**, IX, Theorem 7.9].

We can describe a larger class of rings R and algebras A for which Theorems B and C hold in the following way. We shall say that an R-algebra A is *cohomologically isodimensional* if $R_{\boldsymbol{m}}$-dim $A_{\boldsymbol{m}}=R$-dim A for every maximal ideal $\boldsymbol{m}$ of R. We shall call a commutative ring R *globally isodimensional* provided gl dim $R_{\boldsymbol{m}}=$gl dim R for every maximal ideal $\boldsymbol{m}$ of R.

THEOREM D. *Let R be any commutative noetherian ring of finite global dimension. Let A be an R-progenerator which is an R-algebra of finite cohomological dimension. Assume also that either* (a) *R is globally isodimensional or* (b) *A is cohomologically isodimensional. Then the following hold*:

(1) gl dim $A=R$-dim $A+$gl dim R.
(2) gl dim $A^e=2$ R-dim $A+$gl dim R.
(3) gl dim $A^e=$gl dim R *iff A is R-separable.*
(4) *If R is semilocal*, $\mathrm{hd}_A(A/N)=R$-dim $A+$gl dim $R=$gl dim A.

PROOF. We shall indicate only the proofs of (1) and (4) in the case where R is globally isodimensional, as the rest follow in a similar way. Recall that for any finitely generated algebra A over a noetherian ring R, gl dim $A=\sup[\text{gl dim } A_{\boldsymbol{m}}]$ and gl dim $R=\sup[\text{gl dim } R_{\boldsymbol{m}}]$ where $\boldsymbol{m}$ runs through the maximal ideals of R.

Let $\boldsymbol{m}$ be a maximal ideal of R such that R-dim $A=R_{\boldsymbol{m}}$-dim $A_{\boldsymbol{m}}$. Then

$$\begin{aligned}\text{gl dim } A_{\boldsymbol{m}} &= R_{\boldsymbol{m}}\text{-dim } A_{\boldsymbol{m}} + \text{gl dim } R_{\boldsymbol{m}} \\ &= R\text{-dim } A + \text{gl dim } R\end{aligned}$$

by Theorem B. Hence, gl dim $A \geqq R$-dim $A+$gl dim R. Equality follows from Proposition 1.

For (4), let $\boldsymbol{m}$ be one of the maximal ideals of R and let $J(R)$ denote the Jacobson radical of R. An application of the Chinese Remainder Theorem gives the exact sequence

$$0 \to N \to A \to \prod_{i=1}^{n} A_i/N_i \to 0$$

where n is the number of maximal ideals of R, $\prod A_i$ is the canonical direct product decomposition of $A/J(R)\cdot A$ generated by the idempotents of $R/J(R)$, and N_i is the Jacobson radical of A_i. Result 3 clearly guarantees the separability of A_i/N_i over $R/\boldsymbol{m}_i$.

Since $R_{\boldsymbol{m}}$ is R-flat, we obtain the exact sequence

$$0 \to N_{\boldsymbol{m}} \to A_{\boldsymbol{m}} \to A_j/N_j \to 0$$

where A_j is the summand over $R/\boldsymbol{m}$.

Thus it follows that $N_{\boldsymbol{m}}$ is the Jacobson radical of $A_{\boldsymbol{m}}$. Hence Theorem A gives $\mathrm{hd}_{A_{\boldsymbol{m}}}(A/N)_{\boldsymbol{m}}=\mathrm{hd}_{A_{\boldsymbol{m}}}(A_{\boldsymbol{m}}/N_{\boldsymbol{m}})=R_{\boldsymbol{m}}\text{-dim } A_{\boldsymbol{m}}+\text{gl dim } R_{\boldsymbol{m}}$. Using the well-known result for noetherian algebras that for M finitely generated $\mathrm{hd}_A(M)=\sup[\mathrm{hd}_{A_{\boldsymbol{m}}}(M_{\boldsymbol{m}})]$, the proof now proceeds as in part (1).

Finally, we note the following entertaining

COROLLARY D.1. *Let A be an R-progenerator which is an R-algebra of finite cohomological dimension. Let R be a noetherian ring of finite global dimension. Any two of the following imply the third*:

(a) *A is cohomologically isodimensional*;

(b) *R is globally isodimensional*;

(c) gl dim $A=$gl dim $A_{\boldsymbol{m}}$ *for every maximal ideal $\boldsymbol{m}$ of R.*

We note that the hypothesis of local or semilocal is necessary for Theorems A and D(4) as the 2×2 upper triangular matrix algebra over the rational integers shows. Although the integers are hereditary, the radical of the algebra is easily seen to be projective as a module over the algebra. On the other hand, the rational integers are globally isodimensional; so the algebra has global dimension 2.

In fact, it can be shown that

THEOREM E. *If R is a dedekind domain with infinitely many primes and A is an R-algebra such that A/N is separable and A is an R-progenerator of finite cohomological dimension and if either R is globally isodimensional or A is cohomologically isodimensional, then* R-dim $A=\mathrm{hd}_A(A/N)$.

PROOF. Since N is nilpotent, one can readily verify that A/N is torsion-free and hence projective. Moreover, the separability of A/N guarantees that $(N+\boldsymbol{m}A)/\boldsymbol{m}A$ is the Jacobson radical of $A/\boldsymbol{m}A$. By Results 3 and 4 and the arguments of [7, Theorem 2.1], we have the following chain of equalities:

$$\begin{aligned} R\text{-dim } A &= \sup[R/\boldsymbol{m}\text{-dim } A/\boldsymbol{m}A] = \sup[\mathrm{hd}_{A/\boldsymbol{m}A}(A/(N+\boldsymbol{m}A))] \\ &= \mathrm{hd}_A(A/N), \end{aligned}$$

where the supremum is taken over all maximal ideals $\boldsymbol{m}$ of R. ∎

Propositions 2 and 3 make it clear that the global dimension of R must be finite since there exist algebras which are R-progenerators of every cohomological dimension over any ring R.

It is uncertain what role the hypothesis of noetherian plays, for any local nonnoetherian domain R with a minimal generating set for $\boldsymbol{m}$ consisting of n elements $(x_1, \cdots, x_n)$ such that $(x_1, \cdots, x_i)$ is a prime ideal for $1 \leqq i \leqq n$ has the properties of Theorem A through C provided gl dim R is replaced by n.

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