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A GENERALIZATION OF KOLMOGOROV'S LAW OF THE ITERATED LOGARITHM[1]

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ABSTRACT. A version of the law of the iterated logarithm is proved for sequences of independent random variables which satisfy the central limit theorem in such a way that the convergence of the appropriate moment-generating functions to that of the standard normal distribution occurs at a particular rate. Kolmogorov's law of the iterated logarithm is a corollary of this theorem which, unlike Kolmogorov's result, does not require boundedness of the random variables. Some iterated logarithm results for weighted averages of independent random variables are shown to follow from the main result. Moreover, some applications to sequences of independent, generalized Gaussian random variables are provided.

1. **Introduction.** Let $X_1, X_2, \cdots$ be independent random variables (r.v.), each with mean zero and finite variance (say $EX_n^2=\sigma_n^2$). For $n\geqq 1$ put $S_n=X_1+\cdots+X_n$ and $s_n^2=ES_n^2$. The sequence obeys the law of the iterated logarithm (LIL) if

$$\limsup_{n\to\infty}\frac{S_n}{(2s_n^2\log\log s_n^2)^{1/2}}=1 \quad \text{a.e.} \tag{1}$$

Kolmogorov [**5**, p. 260] proved that (1) holds if $s_n\to\infty$ and $|X_n|s_n^{-1}\leqq a_n=o((\log\log s_n^2)^{-1/2})$. A predictable consequence of this result was that researchers would look for conditions on unbounded r.v. which would ensure the validity of (1). Among the first of such results was one of Hartman [**4**]: if the X_n's are normal with $s_n\to\infty$ and $\limsup \sigma_n/s_n<1$ then (1) holds. The question arises from this theorem as to whether or not a relationship exists between the LIL and the central limit theorem (CLT).

It was known before the appearance of Hartman's work that the CLT may hold while (1) does not (see [**6**]). Moreover, Egorov [**2**] recently

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demonstrated that the validity of (1) does not necessarily entail that of the CLT. Nevertheless, it has been shown (see [**8**] and references mentioned therein) that if the CLT holds and if the convergence in distribution of S_n/s_n to the standard normal is sufficiently rapid, then, indeed, (1) must occur.

Rather than examining the convergence of distribution functions, this paper intends to consider the problem from another angle by examining the convergence of moment-generating functions. The following theorem will elucidate the approach.

THEOREM 1. *Let $X_1, X_2, \cdots$ be a sequence of independent r.v. with $EX_n=0$ and $EX_n^2=\sigma_n^2<\infty$ for all n. Define $S_n=X_1+X_2+\cdots+X_n$ and $s_n^2=ES_n^2$.*

(i) *Suppose B_n denotes a sequence of positive numbers such that $B_n\uparrow\infty$, $B_n\sim B_{n+1}$ (i.e. $B_{n+1}/B_n\to 1$) and $\liminf_{n\to\infty} B_n/s_n>0$. Let $c_n>0$ be a sequence such that $c_n^2 \log\log s_n^2\to 0$. If $\alpha_1, \alpha_2, \cdots$ are positive numbers such that, for all sufficiently large values of n (say $n\geqq N$),*

$$E\exp\{tS_n/(\alpha_n B_n)\} \leqq \exp\{(t^2/2)(1+|t|\,c_n/2)\}$$

whenever $|t|\,c_n\leqq 1$, then

$$\limsup_{n\to\infty} \frac{S_n}{(2B_n^2 \log\log B_n^2)^{1/2}} \leqq \limsup_{n\to\infty} \alpha_n \quad a.e.$$

(ii) *If the hypotheses of* (i) *are fulfilled with $B_n=s_n$ and $\alpha_n\leqq 1$ for $n\geqq N$, and if, in addition,*

$$E\exp\{tS_n/s_n\} \geqq \exp\{(t^2/2)(1-|t|\,c_n)\} \quad \textit{for all } n\geqq N$$

and all t such that $|t|\,c_n\leqq 1$, then (1) *holds.*

The conditions in (ii) of Theorem 1 imply that $E\exp\{tS_n/s_n\}\to\exp\{t^2/2\}$ if $|t|\leqq\min_{n\geqq 1}(c_n^{-1})$, and, hence, that the CLT holds. Since moment-generating functions do not always exist, Theorem 1 may not be as useful as some of the results in [**8**]; but where it is applicable it may turn out to be more easily employed.

Theorem 1 will be proved in §2. §3 considers some consequences and applications of the theorem, while a one-sided LIL result for generalized Gaussian r.v. will be developed in §4.

2. **Proof of Theorem 1.** Like many other LIL results, Theorem 1 will borrow heavily from Kolmogorov's techniques in its proof. To employ such methods, however, exponential bounds are required.

LEMMA 1. *Let S be a r.v.*

(i) *Suppose that, for some $c>0$, and all t such that $0\leqq tc\leqq 1$,*

$$E\exp\{tS\} \leqq \exp\{(t^2/2)(1+tc/2)\}.$$

Then, for any $0<\varepsilon\leqq 1/c$,

$$P[S\geqq\varepsilon]\leqq\exp\{-(\varepsilon^2/2)(1-\varepsilon c/2)\}.$$

(ii) *If, moreover, $E\exp\{tS\}\geqq\exp\{(t^2/2)(1-tc)\}$ whenever $0\leqq tc\leqq 1$ then, given any $\gamma>0$, there exist positive numbers ε_0 and η_0 (depending on γ) such that*

$$P[S\geqq\varepsilon]\geqq\exp\{-(\varepsilon^2/2)(1+\gamma)\} \quad \textit{if} \quad \varepsilon>\varepsilon_0 \quad \textit{and} \quad \varepsilon c<\eta_0.$$

Furthermore, $ES=0$ and $ES^2=1$.

PROOF. To prove (i), note that if $0<\varepsilon c\leqq 1$

$$P[S\geqq\varepsilon]=P[\exp(\varepsilon S)\geqq\exp(\varepsilon^2)]\leqq e^{-\varepsilon^2}E\exp\{\varepsilon S\}.$$

As for (ii), the combined effect of the double inequality involving $E\exp\{tS\}$ is precisely line (1) on p. 255 of [**5**]. By virtually duplicating the proof of Kolmogorov's exponential bounds from that line on, the inequality in (ii) results.

To complete the proof, a technique similar to one used by Stout [**7**, p. 22] will be utilized. Note that, as $t\downarrow 0$,

$$(t/2)(1-tc)+o(1)\leqq ES+o(1)\leqq(t/2)(1+tc/2)+o(1);$$

hence $ES=0$. Using this fact we have, again as $t\downarrow 0$,

$$\tfrac{1}{2}(1-tc)+o(1)\leqq ES^2/2+o(1)\leqq\tfrac{1}{2}(1+tc/2)+o(1),$$

implying $ES^2=1$.

Proceeding now to the proof of Theorem 1, let $\alpha=\limsup\alpha_n$; there is certainly no loss of generality in assuming $\alpha<\infty$. Choose $0<B<\liminf B_n/s_n$. Pick arbitrarily $\delta>\delta'>\delta''>\delta'''>\alpha$ and $1<c<\delta/\delta'$. Let n_k be the least integer m such that $B_m\geqq c^k$; since $B_n\sim B_{n+1}$, we have $B_{n_k}\sim c^k$. For brevity let $t_n^2=2\log\log B_n^2$; note $t_{n_k}\sim t_{n_{k+1}}$ as $k\to\infty$.

Now, there exists a number $K_0>0$ such that, if $k\geqq K_0$,

$$\begin{gathered} n_k>n_{k-1}\geqq N,\quad \delta B_{n_{k-1}}t_{n_{k-1}}>\delta' B_{n_k}t_{n_k}, \\ B_{n_k}\geqq Bs_{n_k},\quad \alpha_{n_k}<\delta''',\quad \delta'-\sqrt{2}\,(Bt_{n_k})^{-1}>\delta'', \\ \delta'' t_{n_k}c_{n_k}\leqq\delta''' \quad\text{and}\quad 1-t_{n_k}c_{n_k}\delta''/(2\delta''')>\delta'''/\delta''. \end{gathered} \tag{2}$$

For each $k\geqq K_0$, let $S_k^*=\max_{n_{k-1}<n\leqq n_k}S_n$.

Using Levy's inequality [**5**, p. 248], Lemma 1(i) and several parts of (2),

one obtains, for $k \geqq K_0$,

$$\begin{aligned}\tfrac{1}{2}P[S_k^* \geqq \delta' B_{n_k} t_{n_k}] &\leqq P[S_{n_k} \geqq \delta' B_{n_k} t_{n_k} - \sqrt{2} s_{n_k}] \\ &\leqq P[S_{n_k} \geqq \delta'' B_{n_k} t_{n_k}] \\ &\leqq P[S_{n_k}/(\alpha_{n_k} B_{n_k}) \geqq (\delta''/\delta''') t_{n_k}] \\ &\leqq \exp\{-(\delta''/\delta''')\log\log B_{n_k}^2\} \\ &\sim [(2\log c)\cdot k]^{-\delta''/\delta'''} \quad \text{as } k \to \infty.\end{aligned}$$

Thus $\sum_1^\infty P[S_k^* \geqq \delta' B_{n_k} t_{n_k}] < \infty$ which implies, by the Borel-Cantelli lemma, that $P[S_k^* \geqq \delta' B_{n_k} t_{n_k} \text{ i.o.}] = 0$. But, again using (2),

$$P[S_n \geqq \delta B_n t_n \text{ i.o.}] \leqq P[S_k^* \geqq \delta B_{n_{k-1}} t_{n_{k-1}} \text{ i.o.}] \leqq P[S_k^* \geqq \delta' B_{n_k} t_{n_k} \text{ i.o.}].$$

Hence part (i) is proved.

Under the conditions of (ii), if $|t|\, c_n \leqq 1$ for any $n \geqq N$, then $(|t|\, \alpha_n) c_n \leqq 1$, which implies that

$$\begin{aligned}E\exp(tS_n/s_n) &\leqq \exp\{(t^2\alpha_n^2/2)(1 + |t|\, \alpha_n c_n/2)\} \\ &\leqq \exp\{(t^2/2)(1 + |t|\, c_n/2)\}.\end{aligned}$$

Therefore no harm is incurred by assuming $\alpha_n = 1$.

Now, by making minor modifications to the proof of the "$\geqq$" part of Theorem 1 of [**8**], and using Lemma 1 where appropriate, the reader may verify part (ii).

3. **Some consequences of the theorem.** Later in this section, some results for weighted averages will be presented. But, first, consider two of the most obvious consequences.

If $X_1, X_2, \cdots$ are $N(0, \sigma_n^2)$ with $\sigma_n/s_n \to 0$ and $s_n \to \infty$, then (1) holds; in this case take $\alpha_n = 1$ and choose c_n arbitrarily in Theorem 1. Note that this corollary is not as strong as Hartman's theorem in [**4**].

Kolmogorov's LIL is implied by Theorem 1; using the notation of §1, take $c_n = a_n$ and $\alpha_n = 1$ for all $n \geqq 1$. That the conditions of the theorem are satisfied is shown by the argument on pp. 254–255 of [**5**].

THEOREM 2. *Let $X_1, X_2, \cdots$ be independent r.v.; suppose $0 < c_n = o((\log\log n)^{-1/2})$. Suppose there is a number $N > 0$ such that, given any $n \geqq N$,*

$$\begin{aligned}\exp\{(t^2/2n)(1 - |t|\, c_n)\} &\leqq E\exp\{n^{-1/2} t X_k\} \\ &\leqq \exp\{(t^2/2n)(1 + |t|\, c_n/2)\}\end{aligned} \tag{3}$$

for each $k \leqq n$ and any t such that $|t|\, c_n \leqq 1$. Then, for each $\alpha \geqq 0$,

$$\limsup_{n\to\infty} n^{-\alpha}(2n\log\log n)^{-1/2} \sum_1^n k^\alpha X_k = (2\alpha + 1)^{-1/2} \quad a.e. \tag{4}$$

PROOF. If $n \geqq N$ and $0 \leqq t\sqrt{n}\, c_n \leqq 1$, then the hypotheses of Lemma 1(ii) are satisfied with $c=\sqrt{n}\, c_n$, and $S=X_k$ $(k \leqq n)$. Hence $EX_k=0$ and $EX_k^2=1$ for all $k \geqq 1$.

Now let $Z_n=\sum_1^n k^\alpha X_k$ and $z_n^2=EZ_n^2=\sum_1^n k^{2\alpha}$. It is evident that $z_n^2 \geqq n^{2\alpha+1}/(2\alpha+1)$; indeed $z_n^2 \sim n^{2\alpha+1}/(2\alpha+1)$. Now define $c_n'=(2\alpha+1)^{1/2}c_n$ and $v_n^2=2 \log \log z_n^2 \sim 2 \log \log n$. Clearly $c_n'=o(v_n^{-1})$.

Suppose $n \geqq N$ and $|t|c_n' \leqq 1$. Defining, for each $k \leqq n$, $t_k=k^\alpha n^{1/2}t/z_n$, then $|t_k|c_n \leqq |t|c_n' \leqq 1$. So, replacing t by t_k in (3) and using independence, it becomes apparent that the conditions of Theorem 1 are satisfied by the sequence $\{n^\alpha X_n\}$, so the desired result follows.

By again referring to the argument on pp. 254–255 of [**5**], the following is immediate from Theorem 2.

COROLLARY 1. *Let $X_1, X_2, \cdots$ be independent r.v., each with mean zero and variance one, such that $|X_n|n^{-1/2} \leqq a_n=o((\log \log n)^{-1/2})$. Then, for any $\alpha>0$, (4) holds.*

As an application of our results to unbounded r.v., the next result is presented.

COROLLARY 2. *Let $X_1, X_2, \cdots$ be independent, identically distributed r.v. with common density $f(x)=(\exp\{-\sqrt{2}\,|x|\})/\sqrt{2}$, $-\infty<x<\infty$ (Laplace distribution). Then (4) is valid for any $\alpha \geqq 0$.*

PROOF. One need only show that (3) holds. By symmetry, t may be assumed positive. The following well-known inequality will be used (see p. 50 of [**3**]):

$$\text{(5)} \qquad \text{if } 0<t<1 \text{ then } \exp(t)<(1-t)^{-1}<\exp\{t/(1-t)\}.$$

It is easily shown that $E \exp(tX_1)=(1-t^2/2)^{-1}$ for all $|t|<\sqrt{2}$.

Let $c_n=n^{-1/4}$, and asume $n \geqq 5$. If $tc_n \leqq 1$ then $t \leqq n^{-1/4}<\sqrt{2}$ and

$$(t/n)(1-t^2/(2n))^{-1}=\frac{t}{n-t^2/2}<\frac{n^{1/4}}{n-n^{1/2}}<n^{-1/4}=c_n.$$

Hence $(1-t^2/(2n))^{-1}=1+(t/2)[(t/n)(1-t^2/(2n))^{-1}]<1+tc_n/2$. But $t^2/2n \leqq (4n)^{-1/2}<1$ so it follows from (5) that the right-hand inequality in (3) holds.

On the other hand, again by (5),

$$(1-t^2/(2n))^{-1}>\exp\{t^2/(2n)\}>\exp\{(t^2/2n)(1-tc_n)\},$$

and the corollary is proved.

4. **A one-sided LIL for generalized Gaussian r.v.** Following Chow [**1**], a r.v. X will be called *generalized Gaussian* if $E \exp\{tX\} \leqq \exp\{t^2\alpha^2/2\}$ for some $\alpha \geqq 0$ and all t. Define $\tau(X)$ to be the smallest α for which the

inequality holds. Chow has noted that, for any a, aX will be generalized Gaussian with $\tau(aX)=|a|\tau(X)$ and that if X and Y are independent and generalized Gaussian, so is $X+Y$ and $\tau^2(X+Y)\leqq\tau^2(X)+\tau^2(Y)$. Furthermore, Stout [7, p. 22] has shown that if X is generalized Gaussian, then $EX=0$ and $EX^2\leqq\tau^2(X)$.

THEOREM 3. *Let $X_1, X_2, \cdots$ be independent generalized Gaussian r.v. Define $\sigma_n^2=EX_n^2$, $S_n=X_1+\cdots+X_n$ and $s_n^2=ES_n^2$. Let $0<B_n\uparrow\infty$, $B_n\sim B_{n+1}$ and suppose* $\liminf B_n/s_n>0$. *If $\alpha_1, \alpha_2, \cdots$ are positive numbers such that $\tau(S_n/(\alpha_n B_n))\leqq 1$, then*

$$\limsup_{n\to\infty}\frac{S_n}{(2B_n^2\log\log B_n^2)^{1/2}}\leqq\limsup_{n\to\infty}\alpha_n.$$

In particular, suppose $s_n\sim s_{n+1}$, $s_n\to\infty$ and $\tau(X_n)\leqq\lambda\sigma_n$ for some $\lambda>0$. Then

$$\limsup_{n\to\infty}\frac{S_n}{(2s_n^2\log\log s_n^2)^{1/2}}\leqq\lambda\quad a.e.$$

PROOF. Clearly $E\exp\{tS_n/(\alpha_n B_n)\}\leqq\exp\{t^2/2\}$ for all t, so that Theorem 1(i) implies the first inequality.

If $\tau(X_n)\leqq\lambda\sigma_n$ for all n, then $E\exp\{tS_n\}\leqq\exp\{\lambda^2t^2s_n^2/2\}$ for all t. So the second inequality is derivable from the first by taking $B_n=s_n$ and $\alpha_n=\lambda$.

The following corollary is due to Chow (Theorem 5 of [**1**]).

COROLLARY 3. *Let $X_1, X_2, \cdots$ be independent, generalized Gaussian r.v. where* $\sup\tau(X_n)<\infty$. *Suppose b_n are real numbers with $b_1>0$, $b_n\geqq 0$. Define $s_n^2=b_1^2+\cdots+n_n^2$ and $B_n=b_1+b_2+\cdots+b_n$ and suppose that $B_n\to\infty$. If $s_n^2/B_n^2=o((\log\log B_n)^{-1})$ (e.g. if $b_n=o(B_n/\log\log B_n)$) then*

$$\lim_{n\to\infty}B_n^{-1}\sum_1^n b_kX_k=0\quad a.e.$$

PROOF. It is apparent that $B_n\sim B_{n+1}$. Define $Q_n=B_n/(\log\log B_n)^{1/2}$; then $Q_n\sim Q_{n+1}$, $\log\log Q_n\sim\log\log B_n$, and $\lim_{n\to\infty}Q_n/s_n=\infty$.

Let $Y_n=\sum_1^n b_kX_k$ and suppose $c\geqq\tau(X_n)$ for all n. Then $E\exp\{tY_n\}\leqq\exp\{c^2t^2s_n^2/2\}$ implying $\tau(Y_n/(cs_n))\leqq 1$.

If X is generalized Gaussian, so is $-X$ with $\tau(-X)=\tau(X)$, so if the conditions of Theorem 3 are satisfied by $X_1, X_2, \cdots$ then they also hold for $-X_1, -X_2, \cdots$. Hence, letting $\alpha_n=cs_n/Q_n$ in Theorem 3, it follows, since $\lim\alpha_n=0$, that

$$\limsup_{n\to\infty}\frac{|Y_n|}{(2Q_n^2\log\log Q_n^2)^{1/2}}=0\quad\text{a.e.}$$

But $Q_n^2\log\log Q_n^2\sim Q_n^2\log\log B_n=B_n^2$, implying the corollary.

REMARK. Corollary 3 implies that, for any sequence X_n of independent generalized Gaussian r.v. with $\sup \tau(X_n) < \infty$, the strong law of large numbers is valid; i.e. $X_1 + \cdots + X_n = o(n)$.

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