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ON SUBALGEBRA LATTICES OF UNIVERSAL ALGEBRAS

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ABSTRACT. If A is a universal algebra, $S(A)$ is the lattice of all subalgebras of A. If $B \subseteq A \times A$, B^* is $\{(x, y):(y, x) \in B\}$.

THEOREM. *Let L_1, L_2, L_3 be algebraic lattices such that $|L_1|$, $|L_2| > 1$. Let α_i be an involutive automorphism of L_i, $i=1, 2$. Then there are two universal algebras A_1, A_2 of the same similarity type, having the properties*:

(a) *there are lattice isomorphisms β_i of L_i onto $S(A_i \times A_i)$, $i=1, 2$, and β_3 of L_3 onto $S(A_1 \times A_2)$*;

(b) $(l\alpha_i)\beta_i = (l\beta_i)^*$, $l \in L_i$, $i=1, 2$.

The lattices of all subalgebras of universal algebras were characterized by G. Birkhoff and O. Frink in **[1]** as algebraic lattices (see also **[2]**, **[3]** for terminology). In **[4]** the author proved that every algebraic lattice is isomorphic to the subalgebra lattice of the square of some universal algebra. A simpler proof of this result, not using the axiom of choice, was given by G. Grätzer and W. A. Lampe in **[3]**.

If A is a universal algebra denote by $\mathscr{S}(A)$ the subalgebra lattice of A. If $B \in \mathscr{S}(A \times A)$ then $B^* = \{(x, y):(y, x) \in B\}$. The aim of this note is to show the following generalization of the main result in **[4]**.

THEOREM. *Let L_1, L_2, L_3 be algebraic lattices such that $|L_1|$, $|L_2| > 1$. Let α_i be an involutive automorphism of L_i, $i=1, 2$. Then there are two universal algebras A_1, A_2 of the same similarity type, having the properties*:

(a) *there are lattice isomorphisms β_i of L_i onto $\mathscr{S}(A_i \times A_i)$, $i=1, 2$, and β_3 of L_3 onto $\mathscr{S}(A_1 \times A_2)$*;

(b) $(l\alpha_i)\beta_i = (l\beta_i)^*$, $l \in L_i$, $i=1, 2$.

The case where one of $|L_1|$, $|L_2|$ is 1 is equivalent to the characterization of the connection between $\mathscr{S}(A)$ and $\mathscr{S}(A \times A)$. This problem for partial universal algebras was solved by the author in **[5]**. For the case of full universal algebras the question remains open.

We will give here the construction of the algebras A_1, A_2. The details of the proof are similar to those of [3] and therefore will be omitted.

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If $A=\langle A; F\rangle$ is a partial algebra, by $\mathscr{S}(A)$ will be denoted the lattice of all (closed) subalgebras of A and by $\mathscr{C}(A)$ the family of all finitely generated subalgebras of A. The set of all subsets of a set X will be denoted by $P(X)$.

LEMMA 1. *Let L_1, L_2, L_3, α_1, α_2 be as in the theorem. Then there are partial algebras $\langle B_1; F\rangle$ and $\langle B_2; F\rangle$ with the properties*:

(i) *there are isomorphisms β_i of L_i onto $\mathscr{S}(B_i\times B_i)$, $i=1, 2$, and β_3 of L_3 onto $\mathscr{S}(B_1\times B_2)$;*

(ii) $(l\alpha_i)\beta_i=(l\beta_i)^*$, $l\in L_i$, $i=1, 2$;

(iii) *all partial operations in F are injective on B_1, B_2, i.e. $f(a_1, \cdots, a_n)=f(b_1, \cdots, b_n)$ implies $a_1=b_1, \cdots, a_n=b_n$;*

(iv) *there are mappings γ_i of $\mathscr{C}(B_i\times B_i)$ into $B_i\times B_i$, $i=1, 2$, and γ_3 of $\mathscr{C}(B_1\times B_2)$ into $B_1\times B_2$ such that every finitely generated subalgebra is generated by one element: namely its image under the appropriate γ_1, γ_2, γ_3.*

Let C_j denote the set of all compact elements in L_j, $j=1, 2, 3$ (consider the L_j mutually disjoint). Let B be an infinite set strictly containing $C_1\cup C_2\cup C_3$. Choose B_1, B_2, two disjoint copies of B. If $b\in B$, denote by b_i the element of B_i corresponding to b (under a fixed bijection from B onto B_i), $i=1, 2$. Since $|L_i|>1$ we have $|C_i|>1$. Choose $c_0^i\in C_i$, $c_0^i\neq 0_i$ (the zero element of L_i) and $c_0^i\alpha_i=c_0^i$, $i=1, 2$. Fix $b_0\in B\setminus(C_1\cup C_2\cup C_3)$.

Define mappings δ_1, δ_2, δ_3, from C_1, C_2, C_3 into $P(B_1\times B_1)$, $P(B_2\times B_2)$ and $P(B_1\times B_2)$ respectively:

For $i=1, 2$,

$$
\begin{aligned}
0_i\delta_i &= \{(b_i, b_i): b\in B\},\\
c\delta_i &= \{(c_i, (c\alpha_i)_i)\} \quad \text{if } c\in C_i,\ c\alpha_i\neq c,\\
c\delta_i &= \{(c_i, b_{0i}), (b_{0i}, c_i)\} \quad \text{if } c\in C_i,\ c\alpha_i = c,\ c\neq 0_i,\ c\neq c_0^i,\\
c_0^i\delta_i &= B_i\times B_i\setminus\bigcup\{c\delta_i : c\in C_i,\ c\neq c_0^i\};\\
c\delta_3 &= \{(c_1, b_{02})\},\ c\in C_3,\ c\neq 0_3,\\
0_3\delta_3 &= B_1\times B_2\setminus\bigcup\{c\delta_3 : c\in C_3,\ c\neq 0_3\}.
\end{aligned}
$$

The following properties of δ_1, δ_2, δ_3 are obvious:

1. $c\delta_j\neq\varnothing$ for all $c\in C_j$, $j=1, 2, 3$.
2. If $c, c'\in C_j$, $c\neq c'$ then $c\delta_j\cap c'\delta_j=\varnothing$, $j=1, 2, 3$.
3. $\bigcup\{c\delta_i : c\in C_i\}=B_i\times B_i$, $i=1, 2$.
4. $\bigcup\{c\delta_3 : c\in C_3\}=B_1\times B_2$.

Define a family $F=F_0\cup F_1\cup F_2\cup F_3$ of partial operations on B_1, B_2 as follows:

To every $b\in B$, associate $f_b\in F_0$—a nullary operation, the result of which, on B_i, is b_i, $i=1, 2$.

To every $c, c', c''\in C_i$ such that each of c, c', c'' is distinct from 0_i and

$c \leqq c' \vee c''$ and to every $(a, b) \in c\delta_i$, $(a', b') \in c'\delta_i$ and $(a'', b'') \in c''\delta_i$, associate a binary partial operation $f \in F_i$ such that

$$f(a', a'') = a, \qquad f(b', b'') = b$$

and

$$D(f, B_i) = \{(a', a''), (b', b'')\}, \qquad D(f, B_k) = \varnothing \quad \text{if } k \neq i,\ i = 1, 2.$$

To every $c, c', c'' \in C_3$ and to every $a, a', a'' \in B_1$ and $b, b', b'' \in B_2$ such that $(a, b) \in c\delta_3$, $(a', b') \in c'\delta_3$ and $(a'', b'') \in c''\delta_3$, associate a binary partial operation $g \in F_3$ such that

$$g(a', a'') = a, \qquad g(b', b'') = b$$

and

$$D(g, B_1) = \{(a', a'')\}, \qquad D(g, B_2) = \{(b', b'')\}.$$

It can be verified that the operations of F are injective on B_1, B_2. The binary operations in F_3 are defined only on $B_1 \times B_2$, on the diagonals of $B_1 \times B_1$ and $B_2 \times B_2$ and nowhere else.

The isomorphisms β_1, β_2, β_3 are defined by

$$l\beta_j = \bigcup \{c\delta_j : c \in C_j,\ c \leqq l\}, \qquad l \in L_j,\ j = 1, 2, 3.$$

It can be shown that $\langle B_1; F\rangle$, $\langle B_2; F\rangle$ satisfy conditions (i) and (ii) of Lemma 1.

Since a subalgebra of a partial algebra A is finitely generated iff it is compact in $\mathscr{S}(A)$, define γ_1, γ_2, γ_3 by:

If $i = 1, 2$,

$$\begin{aligned}
(0_i\beta_i)\gamma_i &= (0_{ii}, 0_{ii}), \\
(c\beta_i)\gamma_i &= (c_i, (c\alpha_i)_i) && \text{if } c \in C_i,\ c\alpha_i \neq c, \\
&= (c_i, b_{0i}) && \text{if } c \in C_i,\ c\alpha_i = c,\ c \neq 0_i, \\
(0_3\beta_3)\gamma_3 &= (0_{31}, 0_{32}), \\
(c\beta_3)\gamma_3 &= (c_1, b_{02}) && \text{if } c \in C_3,\ c \neq 0_3.
\end{aligned}$$

It can be shown that γ_1, γ_2, γ_3 satisfy (iv) of Lemma 1.

If $\boldsymbol{B}$ is a partial algebra, denote by $\boldsymbol{B}^1$ the free algebra generated by $\boldsymbol{B}$; we have:

LEMMA 2. *Let $\langle B_i; F\rangle$, $i = 1, 2$, be partial algebras satisfying all the conditions of Lemma 1. Then*

(i) *the operations of F are injective on B_1^1 and B_2^1;*

(ii) *if $D \in \mathscr{S}(\boldsymbol{B}_i \times \boldsymbol{B}_i)$ and $\bar{D}$ is the subalgebra of $\boldsymbol{B}_i^1 \times \boldsymbol{B}_i^1$ generated by D then $\bar{D} \cap (B_i \times B_i) = D$, $i = 1, 2$;*

(iii) *if $D \in \mathscr{S}(\boldsymbol{B}_1 \times \boldsymbol{B}_2)$ and $\bar{D}$ is the subalgebra of $\boldsymbol{B}_1^1 \times \boldsymbol{B}_2^1$ generated by D then $\bar{D} \cap (B_1 \times B_2) = D$;*

(iv) *$D \to$ the subalgebra of $\boldsymbol{B}_i^1 \times \boldsymbol{B}_i^1$ generated by D is a lattice monomorphism of $\mathcal{S}(\boldsymbol{B}_i \times \boldsymbol{B}_i)$ into $\mathcal{S}(\boldsymbol{B}_i^1 \times \boldsymbol{B}_i^1)$, $i=1, 2$;*

(v) *$D \to$ the subalgebra of $\boldsymbol{B}_1^1 \times \boldsymbol{B}_2^1$ generated by D is a lattice monomorphism of $\mathcal{S}(\boldsymbol{B}_1 \times \boldsymbol{B}_2)$ into $\mathcal{S}(\boldsymbol{B}_1^1 \times \boldsymbol{B}_2^1)$;*

(vi) *if a belongs to the subalgebra of $\boldsymbol{B}_i^1 \times \boldsymbol{B}_i^1$ generated by $B_i \times B_i$ there exists an $\bar{a} \in B_i \times B_i$ such that for $D \in \mathcal{S}(\boldsymbol{B}_i \times \boldsymbol{B}_i)$, a belongs to the subalgebra of $\boldsymbol{B}_i^1 \times \boldsymbol{B}_i^1$ generated by D iff $\bar{a} \in D$, $i=1, 2$;*

(vii) *if a belongs to the subalgebra of $\boldsymbol{B}_1^1 \times \boldsymbol{B}_2^1$ generated by $B_1 \times B_2$, there exists an $\bar{a} \in B_1 \times B_2$ such that for $D \in \mathcal{S}(\boldsymbol{B}_1 \times \boldsymbol{B}_2)$, a belongs to the subalgebra of $\boldsymbol{B}_1^1 \times \boldsymbol{B}_2^1$ generated by D iff $\bar{a} \in D$.*

LEMMA 3. *Let $\langle B_1; F\rangle$, $\langle B_2; F\rangle$ be partial algebras satisfying all the conditions of Lemma* 1. *Then one can define two families of partial operations F', F'' on B_1^1, B_2^1 such that the partial algebras $\boldsymbol{B}_i' = \langle B_i^1; F^1\rangle$, $i=1, 2$, $F^1 = F \cup F' \cup F''$, enjoy all the properties in Lemma* 1, *and properties* (i), (ii) *of Lemma* 2, *where B_i^1 should be replaced by B_i', $i=1, 2$; moreover*

(i) *$D \to$ the subalgebra of $\boldsymbol{B}_i' \times \boldsymbol{B}_i'$ generated by D is a lattice isomorphism of $\mathcal{S}(\boldsymbol{B}_i \times \boldsymbol{B}_i)$ onto $\mathcal{S}(\boldsymbol{B}_i' \times \boldsymbol{B}_i')$, $i=1, 2$;*

(ii) *$D \to$ the subalgebra of $\boldsymbol{B}_1' \times \boldsymbol{B}_2'$ generated by D is a lattice isomorphism of $\mathcal{S}(\boldsymbol{B}_1 \times \boldsymbol{B}_2)$ onto $\mathcal{S}(\boldsymbol{B}_1' \times \boldsymbol{B}_2')$.*

For any a, $\bar{a}$ as in Lemma 2, if $a=(a', a'')$, $\bar{a}=(\bar{a}', \bar{a}'')$ introduce $g \in F'$:

$$D(g, B_1^1) \cup D(g, B_2^1) = \{a', a''\} \qquad \text{and} \qquad g(a') = \bar{a}', \quad g(a'') = \bar{a}''.$$

For any $d=(d', d'')$ not belonging to the subalgebra of $\boldsymbol{B}_i^1 \times \boldsymbol{B}_i^1$ generated by $B_i \times B_i$, $d \in B_i^1 \times B_i^1$, define $h, k \in F''$ by

$$\begin{aligned}
D(h, B_j^1) &= D(k, B_j^1) = \varnothing \quad \text{if } j \neq i,\\
D(h, B_i^1) &= \{d', d''\} \quad \text{and}\\
h(d') &= c_{0i}^i, \qquad h(d'') = b_{0i},\\
D(k, B_i^1) &= \{c_{0i}^i, b_{0i}\} \quad \text{and}\\
k(c_{0i}^i) &= d', \quad k(b_{0i}) = d'', \qquad i = 1, 2.
\end{aligned}$$

For any $d=(d', d'') \in B_1^1 \times B_2^1$, not belonging to the subalgebra of $\boldsymbol{B}_1^1 \times \boldsymbol{B}_2^1$ generated by $B_1 \times B_2$ introduce $h, k \in F''$ by

$$\begin{aligned}
D(h, B_1^1) &= \{d'\}, & D(h, B_2^1) &= \{d''\},\\
h(d') &= 0_{11}, & h(d'') &= 0_{22},\\
D(k, B_1^1) &= \{0_{11}\}, & D(k, B_2^1) &= \{0_{22}\},\\
k(0_{11}) &= d', & k(0_{22}) &= d''.
\end{aligned}$$

It can be verified that $\langle B_i^1; F^1\rangle$, $i=1, 2$, enjoy all the properties of Lemma 3.

Define the sequence of partial algebras $\langle A_{in}; G_n\rangle$, $i=1,2$, $n=0,1,2,\cdots$, by

$$A_{i0} = B_i \quad \text{and} \quad G_0 = F \quad \text{(of Lemma 1)},$$
$$A_{i(n+1)} = (A_{in})^1, \qquad G_{n+1} = (G_n)^1.$$

Put $A_i = \bigcup \{A_{in}: n=0,1,2,\cdots\}$, $i=1,2$, $G=\bigcup\{G_n: n=0,1,2,\cdots\}$. $\langle A_i; G\rangle$, $i=1,2$, are full universal algebras satisfying the conditions (a), (b) of the theorem.

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