AN ELEMENTARY PROOF OF A RADON-NIKODÝM THEOREM FOR FINITELY ADDITIVE SET FUNCTIONS

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Abstract. In 1967 Charles Fefferman proved a Radon-Nikodym theorem for finitely additive measures. We give an elementary proof of a generalization of this theorem.

In this paper we give an elementary proof of the

Theorem. Let \( \mu, \gamma \) be real-valued bounded finitely additive set functions on an algebra \( \Sigma \) of subsets of the set \( S \). Then for every \( \epsilon > 0 \) there exists \( N \in \Sigma \) and a \( \mu \)-simple function \( f \) on \( S \) such that

\[
E \in \Sigma \\& E \subseteq N \Rightarrow |\mu_E| < \epsilon,
\]

\[
E \in \Sigma \\& E \subseteq S \setminus N \Rightarrow |\gamma_E - \int_E f \, d\mu| < \epsilon.
\]

(Hence there is a \( \mu \)-simple function \( g \) with \( E \in \Sigma \Rightarrow |\gamma_E - \gamma(E \cap N) - \int_E g \, d\mu| < \epsilon \).)

If \( \gamma \) is absolutely \( \mu \)-continuous (i.e., for every \( \epsilon > 0 \) there exists \( \delta > 0 \) such that \( |\mu_E| < \delta \Rightarrow |\gamma_E| < \epsilon \)), the Theorem reduces to a result of Fefferman which he proved using functional analysis (see Corollary).

Lemma 1 ("Weak Hahn decomposition"). Let \( \nu \) be a real-valued bounded finitely additive set function on an algebra \( \Sigma \subseteq \text{exp} \, S \). For every \( \alpha > 0 \) there is a \( B \in \Sigma \) (written \( B(\nu, \alpha) \)) such that

\[
E \in \Sigma \\& E \subseteq B \Rightarrow \nu_E > -\alpha, \quad E \in \Sigma \\& E \subseteq S \setminus B \Rightarrow \nu_E < \alpha.
\]

Proof. \( \nu \) is bounded and so there is a smallest natural number \( k \) with \( \nu \leq k \alpha \); for some \( B \in \Sigma \) we have \( \nu_B > k \alpha - \alpha \). If \( E \in \Sigma \) and \( E \subseteq B \), then \( \nu_E = \nu_B - \nu(B \setminus E) > k \alpha - \alpha - k \alpha = -\alpha \); if \( E \in \Sigma \) and \( E \subseteq S \setminus B \), then \( \nu_E = \nu(B \cup E) - \nu_B < k \alpha - (k \alpha - \alpha) = \alpha \).

Lemma 2 (cf. [2, §22.2], [1, Theorem 2]). Let \( \mu, \gamma \) be finite real-valued finitely additive set functions on an algebra \( \Sigma \subseteq \text{exp} \, S \). If there exists \( \delta > 0 \) such that \( \mu > -\delta / 2 \) and \( \gamma > -\delta / 2 \), then there exist \( N \in \Sigma \) and a \( \mu \)-simple function \( f \) on \( S \) such that

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\[ E \in \Sigma \& E \subseteq N \Rightarrow |\mu E| < \delta, \]
\[ E \in \Sigma \& E \subseteq S \setminus N \Rightarrow \gamma E - \int_E f \, d\mu < \delta. \]

**Proof.** Let \( m \) be an integer \( > 2(\gamma S + \delta)(\mu S + \delta)/\delta^2 \). Put

\[ B_k = B \left( \gamma - \frac{\delta k}{2(\mu S + \delta)} , \frac{\delta}{2m} \right) \quad \text{(by Lemma 1)}, \]

\[ A_1 = S \setminus \bigcup_{j=1}^{m} B_j, \quad A_k = B_{k-1} \setminus \bigcup_{j=k}^{m} B_j, \quad k = 2, 3, \ldots, m, \]

\[ N = B_m, \quad f = \sum_{k=1}^{m} \frac{\delta(k-1)}{2(\mu S + \delta)} \chi_{A_k}, \]

where \( \chi_A \) is the characteristic function of \( A \). If \( E \in \Sigma \) and \( E \subseteq N \), then

\[ \gamma E - \frac{\delta m}{2(\mu S + \delta)} \mu E > - \frac{\delta}{2m} \geq - \frac{\delta}{2}; \]

thus

\[ \mu E < \frac{2}{m} \frac{\mu S + \delta}{\delta} \left( \gamma E + \frac{\delta}{2} \right) < \frac{\delta^2}{(\gamma S + \delta)(\mu S + \delta)} \frac{\mu S + \delta}{\delta} \times \left( \gamma S - \gamma(S|E) + \frac{\delta}{2} \right) < \frac{\delta}{\gamma S + \delta} \left( \gamma S + \gamma S + \frac{\delta}{2} + \frac{\delta}{2} \right) = \delta. \]

If \( E \in \Sigma \) and \( E \subseteq S \setminus N \), then \( E \) is the disjoint union of the sets \( E_1, \ldots, E_m \), where \( E_k = E \cap A_k \) for \( k = 1, 2, \ldots, m \). Moreover, since \( E_k \subseteq A_k \subseteq S \setminus B_k \), we have

\[ \gamma E_k - \frac{\delta k}{2(\mu S + \delta)} \mu E_k < \frac{\delta}{2m} \quad \text{for } k = 1, 2, \ldots, m, \]

and since \( E_k \subseteq A_k \subseteq B_{k-1} \), we have

\[ \gamma E_k - \frac{\delta(k-1)}{2(\mu S + \delta)} \mu E_k > - \frac{\delta}{2m} \quad \text{for } k = 2, \ldots, m. \]

Hence

\[ -\delta < - \frac{\delta}{2} + (m-1) \left( - \frac{\delta}{2m} \right) \leq \sum_{k=1}^{m} \left( \gamma E_k - \frac{\delta(k-1)}{2(\mu S + \delta)} \mu E_k \right) \]

\[ = \sum_{k=1}^{m} \left( \gamma E_k - \frac{\delta k}{2(\mu S + \delta)} \mu E_k \right) + \sum_{k=1}^{m} \frac{\delta}{2(\mu S + \delta)} \mu E_k \]

\[ < m \frac{\delta}{2m} + \frac{\delta}{2} \frac{\mu E}{\mu S + \delta} < \delta. \]
Since $\sum_{k=1}^{m} (\gamma E_k - (\delta(k-1)/2(\mu S + \delta)) \mu E_k) = \gamma E - \int_E f \, d\mu$, Lemma 2 is proved.

**Proof of the Theorem.** Let $B=B(\mu, \varepsilon/8)$, $B'=B(\gamma, \varepsilon/8)$, $\mu_1 E = \mu(E \cap B)$, $\mu_2 E = -\mu(E \setminus B)$, $\gamma_1 E = \gamma(E \cap B')$ and $\gamma_2 E = -\gamma(E \setminus B')$ for $E \in \Sigma$ (so that $\mu = \mu_1 - \mu_2$ and $\gamma = \gamma_1 - \gamma_2$). Clearly $\mu_1 > -\varepsilon/8$ and $\gamma_1 > \varepsilon/8$ for $i=1, 2$. By Lemma 2 (with $\delta = \varepsilon/4$) for $i,j=1, 2$ there are sets $N_{i,j}$ and $\mu$-simple functions $f_{i,j}$ such that

$$E \in \Sigma \land E \subseteq N_{i,j} \Rightarrow |\mu E| < \frac{\varepsilon}{4},$$

$$E \in \Sigma \land E \subseteq S \setminus N_{i,j} \Rightarrow |\gamma E - \int_E f_{i,j} \, d\mu| < \frac{\varepsilon}{4}.$$

Let $N = \bigcup_{i,j=1,2} N_{i,j}$ and $f = (f_{11} - f_{12}) \chi_B + (f_{21} - f_{22}) \chi_{S \setminus B}$. If $E \in \Sigma$ and $E \subseteq N$, then $E$ is the disjoint union of sets $E_{i,j}$, $i,j=1, 2$, with $E_{i,j} \in \Sigma$ and $E_{i,j} \subseteq N_{i,j}$,

$$\mu E = \sum_{i,j=1,2} \mu E_{i,j} \quad \text{and} \quad -\varepsilon < 4 \cdot \left( -\frac{\varepsilon}{8} \right) = \sum_{i,j=1,2} \mu E_{i,j} < 4 \cdot \frac{\varepsilon}{4} = \varepsilon,$$

i.e., $|\mu E| < \varepsilon$. If $E \in \Sigma$ and $E \subseteq S \setminus N$, then $E \subseteq S \setminus N_{i,j}$ for $i,j=1, 2$; in addition $\mu_2(E \cap B) = \mu_2(E \setminus B) = 0$, and thus

$$|\gamma E - \int_E f \, d\mu| \leq |\gamma(E \cap B) - \int_{E \cap B} (f_{11} - f_{12}) \, d\mu|$$

$$+ |\gamma(E \setminus B) - \int_{E \setminus B} (f_{21} - f_{22}) \, d\mu|$$

$$\leq |\gamma_1(E \cap B) - \int_{E \cap B} f_{11} \, d\mu|$$

$$+ |\gamma_2(E \setminus B) - \int_{E \setminus B} f_{12} \, d\mu|$$

$$+ |\gamma_1(E \setminus B) - \int_{E \setminus B} f_{21} \, d\mu|$$

$$+ |\gamma_2(E \setminus B) - \int_{E \setminus B} f_{22} \, d\mu| < 4 \cdot \frac{\varepsilon}{4} = \varepsilon.$$

The Theorem is proved.

It is now easy to prove the

**Corollary.** Let $\mu, \gamma$ be bounded real-valued finitely additive set functions on an algebra $\Sigma \subseteq \exp S$. Let $\gamma$ be absolutely $\mu$-continuous (in the
sense mentioned above). Then for every $\varepsilon > 0$ there is a $\mu$-simple function $f$ such that

$$E \in \Sigma \Rightarrow \left| \gamma E - \int_E f \, d\mu \right| < \varepsilon.$$  

As Fefferman [1, Lemma 1] has shown, it suffices to suppose that $\gamma$ is finite (instead of bounded).

By the Corollary (if $\mu$ and $\gamma$ satisfy its hypotheses), there is a sequence $\{f_n\}$ of $\mu$-simple functions $f_n$ with $\lim_{n \to \infty} \int_E f_n \, d\mu = \gamma E$ uniformly for $E \in \Sigma$ (it was in this form formulated in [1]).

The following example shows there need not be an increasing such sequence:

$\mathcal{S}$ is the set of all natural numbers ($\geq 1$),
$E$ is the algebra of all finite subsets of $\mathcal{S}$ and their complements,

$$\gamma E = 0, \quad \mu E = \sum_{k \in E} 2^{-k} \quad \text{for finite } E,$$

$$\gamma E = 1, \quad \mu E = 1 + \sum_{k \in E} 2^{-k} \quad \text{for infinite } E.$$

References


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